The Kakeya Problem

Edward Kroc
The University of British Columbia
August 2010

Acknowledgements

I am forever grateful and indebted to Malabika Pramanik for her persistent guidance through the subject matter and for always happily sharing her mathematical expertise. No part of this essay would have been possible if not for her tireless patience and constant encouragement.

Chapter 1

Introduction and Definitions

In 1917, Soichi Kakeya proposed the following geometric problem [19]: *find a figure of the least area on which a needle of length* 1 *can be turned continuously through* 360^o. Solved soon thereafter by Besicovitch [4], this seemingly innocuous geometric query became the launching pad for what is known today as the Kakeya problem. At its core, the modern version of the problem asks just how small certain sets in $\mathbb{R}^n$ can be. Before we can make this matter more precise though, we must first define what we mean by a Kakeya set.

Unless otherwise stated, we will always work in two dimensions or higher so that when we speak of $\mathbb{R}^n$ we require $n \geq 2$. Generally speaking then, *a Kakeya set is simply any set which contains a unit line segment in every direction*[1]. Clearly, there are many familiar examples of such a set, the most obvious of which is surely the unit n-ball. In 1920 however, Besicovitch [3] gave an example that was far from familiar, one of a Kakeya set in the plane of Lebesgue measure zero. Perhaps a noteworthy curiosity at the time, over the course of the last forty years Besicovitch's example has proven to be of paramount interest, becoming a touchstone of modern analytical research.

Besicovitch was originally interested in solving the following problem in Riemann integration: *given a Riemann-integrable function f on $\mathbb{R}^2$, must there exist a rectangular coordinate system (x, y) such that $f(x, y)$ is a Riemann-integrable function of x for each fixed y, and the two-dimensional integral of f is equal to the iterated integral $\int\int f(x, y)dxdy$?* The solution to this problem required Besicovitch to construct *a set of segments of length* 1 *of all directions, such that its Jordan* [Lebesgue] *measure is zero.* It was later pointed out to Besicovitch that his solution could be easily modified to resolve the Kakeya needle problem. In 1926, Besicovitch [4] published this solution along with the first explicit constrution of a Kakeya set with Lebesgue measure zero[2].

We note that once we have a Kakeya set in the plane with measure zero, we immediately have Kakeya sets in any $\mathbb{R}^n$ with measure zero just by taking the direct product

[1] Note that any requirement of "continuous movement" of these segments has been dropped.

[2] See [24] for further history.

of the former set and any unit cube of $n-2$ sides. In many ways, these sets are higher dimensional analogues of the classical Cantor set, but as we shall see the true structure of the Kakeya set lies within its geometry. Additionally, a better knowledge of these sets will afford us a deeper understanding of dimensionality in general.

Besicovitch's original construction involves a meticulous dissection and translation of triangles. Many alternate constructions have since been devised; we refer the reader to [27] and [32] for two of the most classical. We will not spend time on an explicit construction ourselves since the options are well documented in the literature already.

1.1 Kakeya Set Conjectures

In what follows, S^{n-1} will always denote the set of directions or unit rays planted at the origin in $\mathbb{R}^n$. We take the following as our formal definition of a Kakeya set.

Definition 1.1.1. *A* ***Kakeya set*** *is a Borel set $K \subset \mathbb{R}^n$ that contains a unit line segment in every direction, i.e.*

$$\forall\ e \in S^{n-1}\ \exists\ a \in \mathbb{R}^n : a + te \in K\ \forall\ t \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

For obvious reasons, Kakekya sets are also referred to as Besicovitch sets in much of the literature, but we will use the terminology of the above definition for this exposition.

As mentioned earlier, the main problem is to decide how small a Kakeya set can be. Of course, we need to make precise what we mean by "how small." There are many ways to do this in terms of various fractal dimensions, but we will concern ourselves with the most classical version in terms of the Hausdorff dimension. For completeness, we will provide a brief introduction to the subject of Hausdorff dimension that will prove sufficient for our purposes.

For any $d \in \mathbb{R}$, $\epsilon > 0$, $E \subset \mathbb{R}^n$, define

$$H_d^\epsilon(E) = \inf \left\{ \sum_{j=1}^{\infty} r_j^d : E \subseteq \bigcup_{j=1}^{\infty} B(x_j, r_j),\ r_j < \epsilon \right\},$$

where $B(x_j, r_j)$ denotes the ball centered at $x_j \in \mathbb{R}^n$ of radius r_j. Notice that the infimum is taken over all countable coverings of E by unions of balls with radii less than ϵ. Notice also that as $\epsilon \to 0$, $H_d^\epsilon(E)$ is nondecreasing by definition of the infimum. Hence the limit exists and we define

$$H_d(E) = \lim_{\epsilon \to 0} H_d^\epsilon(E)$$

to be the *d-dimensional Hausdorff measure* of the set E.

With an eye to technicality, notice that H_d is really just an outer measure. Using the standard theory however, we know that H_d is in fact an honest measure on the Borel subsets of $\mathbb{R}^n$ (see [16] for example).

We wish to define the *Hausdorff dimension* of the set E; to do this, we need to establish a few basic facts.

Fact 1.1.2. *$H_d(E)$ is a nonincreasing function of d.*

Proof. Suppose $d_1 < d_2$ and $\epsilon \leq 1$. Then $H^\epsilon_{d_2}(E) \leq H^\epsilon_{d_1}(E)$, verifying the fact. □

Fact 1.1.3. *If $E \subseteq F$, then $H_d(E) \leq H_d(F)$.*

Proof. Since $E \subseteq F$, any cover of F will also be a cover of E. The result then follows by the definition of infimum. □

Fact 1.1.4. *If $d > n$, then $H_d(E) = 0$ for all $E \subseteq \mathbb{R}^n$.*

Proof. In light of Fact 1.1.3, we fix $d > n$ and aim to show that $H_d(\mathbb{R}^n) = 0$.

Rather than work with covers consisting of balls of certain radii, we will consider covers comprised of cubes with a certain side length, s. The volume of any such cube is clearly equal to the volume of a ball of radius s up to some constant multiple depending only on n.

Fix $\epsilon > 0$ and let j_0 be such that $C_{n,d} \cdot 2^{-j_0(d-n)} < \epsilon$ where the constant $C_{n,d}$ will be determined later. Let $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and consider the corresponding "integer box"

$$\prod_{i=1}^{n} [m_i, m_i + 1).$$

For any $\bar{m}$, we can clearly cover its integer box with an essentially disjoint union of dyadic cubes, each with side length 2^{-j} for any $j \geq 0$. Let $\{\bar{m}_0, \bar{m}_1, \ldots\}$ be an enumeration of $\mathbb{Z}^n$ and for any $\bar{m}_i$ we cover the corresponding integer box by dyadic cubes of side length $2^{-(j_0+i)}$. Notice that the number of such cubes needed to do this is precisely $2^{n(j_0+i)}$; consequently, we have a countable covering of $\mathbb{R}^n$. Let $\{E_1, E_2, \ldots\}$ be an enumeration of this covering and let s_k denote the side length of the kth cube. Then

$$\sum_k s_k^d = \sum_{j=j_0}^{\infty} 2^{nj} \cdot 2^{-dj} = \sum_{j=j_0}^{\infty} \left(2^{n-d}\right)^j = \frac{2^{-j_0(d-n)}}{1 - 2^{n-d}},$$

since $d > n$ implies that $2^{n-d} < 1$. Setting $C_{n,d} = (1 - 2^{n-d})^{-1}$, we see that the above expression is less than ϵ. Thus

$$H^\epsilon_d(\mathbb{R}^n) \lesssim \sum_k s_k^d < \epsilon$$

and letting $\epsilon \to 0$ gives $H_d(E) = 0$ for all $E \subseteq \mathbb{R}^n$. □

As in the proof above, we will always use the notation $A \lesssim B$ to mean that there exists some absolute constant C such that $A \leq C \cdot B$. Sometimes we will also write $A \lesssim_r B$ to emphasize that the implicit constant depends on the fixed parameter r. Similarly, we write $A \gtrsim B$ to mean that there exists some absolute constant C such that $A \geq C \cdot B$. The notation $A \sim B$ will signify that both $A \lesssim B$ and $A \gtrsim B$ hold. This notation is extremely convenient in what follows because we often do not need to keep track of the constants from line to line; we usually only require the knowledge that this constant does not depend on the variable parameters of the objects in question.

After one more fact, we will be ready to define the Hausdorff dimension of a set.

Fact 1.1.5. *To any $E \subseteq \mathbb{R}^n$ there exists a unique $d_0 \leq n$ such that $H_d(E) = \infty$ for all $d < d_0$ and $H_d(E) = 0$ for all $d > d_0$.*

Proof. Clearly, $H_0(E) = \infty$ so define $d_0 = \sup\{d : H_d(E) = \infty\}$. By Fact 1.1.4, $d_0 \leq n$ and since $H_d(E)$ is nonincreasing in d, we have that $H_d(E) = \infty$ for all $d < d_0$. Now suppose that $d > d_0$ and let $\beta \in (d_0, d)$. Clearly, $H_\beta(E) < \infty$. So by definition, if $\epsilon > 0$, then we can find a covering of E by balls of radii $r_j < \epsilon$ with $\sum_j r_j^\beta \leq 1 + H_\beta(E)$. Thus,

$$\sum_j r_j^d = \sum_j r_j^{(d-\beta)+\beta} < \epsilon^{d-\beta} \sum_j r_j^\beta \leq \epsilon^{d-\beta} \left(1 + H_\beta(E)\right) \longrightarrow 0 \quad as \quad \epsilon \to 0.$$

Therefore, $H_d(E) = 0$. □

Definition 1.1.6. *The d_0 defined in Fact 1.1.5 is called the **Hausdorff dimension** of the set $E \subset \mathbb{R}^n$ and is generally denoted by $\dim_H(E)$.*

With this definition in hand, we can now formally state the Kakeya conjecture.

Conjecture 1.1.7 (Kakeya Set Conjecture)**.** *If $K \subset \mathbb{R}^n$ is a Kakeya set, then $\dim_H(K) = n$.*

For $n = 2$, the conjecture has been confirmed since 1971. This is originally due to Davies [12] and we will prove the result ourselves in Chapter 4, although our approach will follow a more modern route. For $n \geq 3$, the conjecture remains open.

The Hausdorff dimension offers a way to quantify the size of a set in $\mathbb{R}^n$, just as Lebesgue measure does. We will say a set in $\mathbb{R}^n$ has full dimension if its Hausdorff dimension is exactly n. Any set of positive Lebesgue measure necessarily has full dimension but it is also true that zero Lebesgue measure sets can have full dimension. The most common examples of such sets are certain fractal sets (e.g. Julia sets or Peano curves) and we refer the reader to Falconer's excellent text on the subject [14]. The terminology of "fullness" is a good one when discussing dimensionality, both conceptually and to distinguish the idea from that of the "volume" of a set with respect to Lebesgue measure.

What this means for Kakeya is the following. From Fact 1.1.4, we know that $\dim_H(E) \leq n$ for any $E \subset \mathbb{R}^n$, including the case when E is Kakeya. The Kakeya conjecture asserts that Kakeya sets have full dimension. So even though these sets may have Lebesgue measure zero, they still fill out enough of $\mathbb{R}^n$ to not lose anything in terms of dimension. We can draw an analogy from the realm of physics to put the situation in a more intuitive, albeit heuristic, perspective. Matter is classically defined to mean anything that has mass and occupies volume. Naturally we think of the Lebesgue measure of a set as its volume and we can consider its Hausdorff dimension to be a quantification of its mass. A zero Lebesgue measure Kakeya set K is not made of matter (it has no volume), but if the set conjecture were true, then we would know that K still has mass, in fact it has as much mass as the unit n-ball. In this way, the set K would appear similar to a system of photons with so much energy as to be equivalent to the mass of an actual rubber ball. This would be the content of the full conjecture. Obviously, partial progress on the conjecture would come in the form of a lower bound for $\dim_H(K)$, K a Kakeya set, and these types of lower bounds are precisely what we will concern ourselves with for the majority of this exposition.

This is all very nebulous of course, and this philosophical approach is by no means the only motivation for studying the Kakeya problem. A better understanding of Kakeya means a richer and more powerful analytical arsenal is at our disposal when attempting to analyze functions and operators defined on submanifolds in $\mathbb{R}^n$. As we will see in Chapter 8, the Kakeya problem is intimately related to the question of Bochner-Riesz summability and to the restriction conjecture, both of which tackle problems in analysis of objects lying in a larger dimensional space.

Although we will not dwell on the matter here, another formulation of the Kakeya problem is in terms of the Minkowski dimension of a set.

Definition 1.1.8. *Let $E \subset \mathbb{R}^n$, $\epsilon > 0$. Let $N_\epsilon(E)$ denote the smallest number of cubes of side-length ϵ that cover E. Define the* ***Minkowski dimension*** *of E to be*

$$\dim_M(E) = \lim_{\epsilon \to 0} \frac{\log N_\epsilon(E)}{\log(1/\epsilon)}.$$

If this limit does not exist, then we can only define the ***upper and lower Minkowski dimensions*** *corresponding to the limit superior and limit inferior respectively in the above expression.*

We note that there are many equivalent formulations of the Minkowski dimension of a set; see [14] for a thorough discussion.

Conjecture 1.1.9. *If $K \subset \mathbb{R}^n$ is a Kakeya set, then* $\dim_M(K) = n$.

It is well known that the upper and lower Minkowski dimensions of a set in $\mathbb{R}^n$ majorize its Hausdorff dimension so that lower bounds for the latter imply lower bounds for the former; hence this conjecture too is settled for $n = 2$. We will not say much more about the Minkowski dimension formulation of the Kakeya problem; the current best results for the upper Minkowski dimension are due to Katz, Łaba and Tao [20].

1.2 The Kakeya Maximal Function

There is a third, more quantitative formulation of the Kakeya problem in terms of maximal functions. It is this formulation that we will work with throughout the remainder of this essay as it is not only the most natural thing to study from an analytical perspective, but also because maximal function estimates imply Hausdorff dimension estimates as we will show in Chapter 2.

Maximal functions are essentially averaging operators; they average a function over a certain collection of domains and then take the largest of these averages. The collection of domains that we will be concerned with are thin tubes which we can formally define as follows.

For any $\delta > 0$, $e \in S^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot e| \le \frac{1}{2},\ |\mathrm{proj}_{e^\perp}(x-a)| \le \delta\}, \tag{1.1}$$

where $e^\perp$ is the hyperplane through the origin with e as normal vector. This defines $T_e^\delta(a)$ to be a tube centered at a of unit length oriented in the e direction with cross-sectional radius δ. We will always take $\delta \ll 1$ and will often omit the center of the tube from the notation. As we will see, for many applications it is the orientation and the thickness of the tube that will matter for our purposes, not the exact location in space. We will also use the notation $C \cdot T_e^\delta(a)$ to denote the $(C \times C\delta \times \cdots \times C\delta)$-tube centered at a, oriented in the e direction. Usually, C will denote some constant, but clearly the notation makes sense for any numerical quantity C that we specify.

We should elaborate here on the representation given in (1.1). Notice that if we choose an orthonormal basis for $e^\perp$, say $\zeta_1, \ldots, \zeta_{n-1}$, then for any $x \in T_e^\delta(a)$ we can write

$$x - a = \alpha e + \sum_{i=1}^{n-1} \alpha_i \zeta_i \tag{1.2}$$

where $|\alpha| \le \frac{1}{2}$ and $\left(\sum_{i=1}^{n-1} |\alpha_i|^2\right)^{1/2} \le \delta$. We will often only need the fact that $|\alpha_i| \le \delta$, but it is important to keep equation (1.1) and the true conditions on the parameters α_i in mind. Also note that for $x \in T_e^\delta(a)$, $|(x-a)\cdot v| \le \delta$ for all unit vectors v orthogonal to e if and only if $|\mathrm{proj}_{e^\perp}(x-a)| \le \delta$. This fact, along with the representation in (1.2), will become quite important later, specifically in Chapters 3 and 6.

These tubes can be viewed as essentially δ-neighborhoods of a unit line segment and it is this interpretation that makes studying such objects natural in the context of Kakeya sets. We will often speak of this unit line segment as the "principal axis" of a tube $T_e^\delta(a)$ and we remark here that this will always be identified as the unit line segment given by the intersection of the unique line through a oriented in the e direction with the tube $T_e^\delta(a)$.

Now imagine taking a δ-neighborhood of a Kakeya set $K \subset \mathbb{R}^n$. Then there must be a subset of this neighborhood K_δ that will be a union of precisely these tubular objects just defined. Now consider the characteristic function of this δ-neighborhood, i.e. the function χ_{K_δ} that is equal to 1 on K_δ and is 0 otherwise. If we integrate this function, we get a measure of the "fullness" of this set. Heuristically, if we understand how this fullness changes as we let $\delta \to 0$, then we can draw certain conclusions about the fullness of the actual Kakeya set K. We will show precisely how this can be done in the next chapter with the aid of the maximal function that we now define. Note that all mention of L^p spaces are with respect to the Lebesgue measure on $\mathbb{R}^n$ or with respect to the surface measure on S^{n-1} induced by the Lebesgue measure on $\mathbb{R}^n$. Which one we mean will be obvious from the context.

Definition 1.2.1. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and define the **Kakeya maximal function** $f_\delta^* : S^{n-1} \to \mathbb{R}$ via*

$$f_\delta^*(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f(y)| dy. \tag{1.3}$$

Equivalently, we will often write

$$f_\delta^*(e) = \sup_{T \| e} \frac{1}{|T|} \int_T |f(y)| dy, \tag{1.4}$$

where the supremum is taken over all tubes T with length 1 in the e direction and width δ in all orthogonal directions.

A few words about this definition are in order since this will be the main object of our analysis in the subsequent pages. First, the requirement that $f \in L^1_{loc}(\mathbb{R}^n)$ is to ensure that f_δ^* is well-defined; in practice, we will always be interested in functions from the much smaller L^p spaces, $1 \leq p \leq \infty$. Secondly, the maximal function is a supremum of averaging operators over thin tubes pointing in a given direction, so in particular it is *not* a linear operator. The location of these tubes in space is immaterial since the operation of integration requires us to average only over those parts of the tubes that intersect the support of f. It is clear that equations (1.3) and (1.4) are consistent and we will usually prefer to use (1.4) for the sake of brevity. Unless otherwise stated, all unspecified tubes T will always be $(1 \times \delta \times \cdots \times \delta)$-tubes in $\mathbb{R}^n$.

As mentioned prior to the definition, the maximal function should give us a way to measure the fullness of a Kakeya set, at least in some sense. To do this, we will have to consider the maximal function as an operator acting on functions in some L^p space[3]. We will be interested in majorizing the norm of this Kakeya operator by some quantity that will depend on the size of the space n, the size of the tubes we average over δ, and (presumably) the value of p. This will be the substance of the third and strongest conjecture, the one with which we will occupy most of our time.

Conjecture 1.2.2 (Kakeya Maximal Function Conjecture)**.**

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^*||_{L^n(S^{n-1})} \leq C_\epsilon \delta^{-\epsilon} ||f||_{L^n(\mathbb{R}^n)}. \tag{1.5}$$

Notice that the estimate (1.5) puts a majorant on the norm of the maximal function as an operator from L^n to L^n and that this n is the same as the dimension of the space $\mathbb{R}^n$. The assertion of the conjecture may feel artificial and unmotivated, but as we will see from the next proposition, this is indeed the reasonable thing to expect.

En route to the estimate (1.5), we will study estimates of the form

$$||f_\delta^*||_{L^q(S^{n-1})} \leq \mathbf{K}_\delta(p,n) ||f||_{L^p(\mathbb{R}^n)} \tag{1.6}$$

where $\mathbf{K}_\delta(p,n)$ is a constant depending only on p and n for any fixed $\delta \ll 1$. In this notation, we see that the maximal function conjecture is simply that $\mathbf{K}_\delta(p,n) = O(\delta^{-\epsilon})$ for all $\epsilon > 0$. Usually we will take $q = p$. Hypothetically, the constant $\mathbf{K}_\delta(n,p)$ could depend on the fixed parameter q as well, although in practice this is never the case, as we shall see.

Proposition 1.2.3.

1. *The Kakeya operator is bounded as a sublinear operator mapping $L^\infty(\mathbb{R}^n) \to L^\infty(S^{n-1})$ and $L^1(\mathbb{R}^n) \to L^\infty(S^{n-1})$. In fact,*

$$||f_\delta^*||_\infty \leq ||f||_\infty \tag{1.7}$$

and

$$||f_\delta^*||_\infty \lesssim \delta^{-(n-1)} ||f||_1 \tag{1.8}$$

2. *For any $1 \leq q \leq \infty$, if $p < \infty$, there can be no bound of the form*

$$||f_\delta^*||_q \leq C ||f||_p \tag{1.9}$$

with C independent of δ.

[3] The notation for the maximal function given above (due to Bourgain [6]) is standard but unfortunate when we view it as an operator. We will often have occasion to speak of the norm of this operator from one L^p space to another. For this reason, when we speak of the maximal function as an operator we will not use the standard shorthands to avoid confusion between conventional understanding of notation in operator theory and our own notation.

Proof. The L^1 and L^∞ bounds follow by making the obvious estimates. For the second part of the proposition, let K be a Kakeya set in $\mathbb{R}^n$, K_δ be its δ-neighborhood, and let $f = \chi_{K_\delta}$. Then $f^*_\delta(e) = 1$ for all $e \in S^{n-1}$; consequently, for any $1 \leq q \leq \infty$ we have $||f^*_\delta||_q \sim 1$. Now let $p < \infty$ and compute $||f||_p = |K_\delta|^{\frac{1}{p}}$. If we have an estimate as in (1.7), then this example gives

$$1 \lesssim C|K_\delta|^{\frac{1}{p}}.$$

By definition, we have $\lim_{\delta\to 0}|K_\delta| = 0$, so in order for the above estimate to hold, it must be that $C \to \infty$ as $\delta \to 0$. □

This proposition suggests that the ideal bound in (1.6) should take the form $\mathbf{K}_\delta(p, n) = O(\delta^{-\alpha})$ for some $\alpha > 0$. In fact, we see now that the conjectured bound $\mathbf{K}_\delta(p, n) = O(\delta^{-\epsilon})$ for all $\epsilon > 0$ would be truly optimal. Further evidence for this will come in Chapter 4 when we show Córdoba's L^2 estimate which gives precisely this bound in $\mathbb{R}^2$. But perhaps more suggestive is the proposition in Section 2.3 that asserts if indeed $\mathbf{K}_\delta(p, n) = O(\delta^{-\epsilon})$ for all $\epsilon > 0$, then $\dim_H(K) = n$ for any Kakeya set $K \subset \mathbb{R}^n$. Before we begin this program of study however, it is prudent to mention a problem very similar to Kakeya, that of Nikodym sets.

1.3 The Nikodym Maximal Function

Closely related to Kakeya sets is the problem of Nikodym sets. Both these objects became a subject of study around the same time and with good reason. The two objects are intimately related as we will see in more detail in Chapter 3.

Definition 1.3.1. *A* ***Nikodym set*** *is a Borel set $N \subset \mathbb{R}^n$ such that for every $a \in \mathbb{R}^n$ there exists a line l through a such that $N \cap l$ contains a unit line segment.*

Just as with Kakeya, we have the following conjectures.

Conjecture 1.3.2. *If $N \subset \mathbb{R}^n$ is a Nikodym set, then $\dim_H(N) = n$.*

Conjecture 1.3.3. *If $N \subset \mathbb{R}^n$ is a Nikodym set, then $\dim_M(N) = n$.*

There is also a Nikodym maximal function.

Definition 1.3.4. *Let $f \in L^1_{loc}(\mathbb{R}^n)$ and define the* ***Nikodym maximal function*** *$f^{**}_\delta : \mathbb{R}^n \to \mathbb{R}$ via*

$$f^{**}_\delta(x) = \sup_{T^\delta \ni x} \frac{1}{|T^\delta|}\int_{T^\delta} |f(y)|dy,$$

where the supremum is taken over all $(1 \times \delta \times \cdots \times \delta)$-tubes that contain the point x.

Similarly, there is a maximal function conjecture.

Conjecture 1.3.5 (Nikodym Maximal Function Conjecture)**.**

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^{**}||_{L^n(\mathbb{R}^n)} \leq C_\epsilon \delta^{-\epsilon} ||f||_{L^n(\mathbb{R}^n)}. \tag{1.10}$$

At first glance, it may appear that the Nikodym maximal function is actually the more natural one to work with when studying the size of Kakeya sets; indeed, this is the function Córdoba [11] first applied to the analysis. The Kakeya maximal function enjoys more convenient analytical properties though and so has become the main object of interest over time. It is clear that Nikodym and Kakeya sets share similar bits of dimensional oddity, but what is truly amazing is that they hold essentially the same oddities. We will see in Chapter 3 that the Nikodym and Kakeya set conjectures are actually equivalent. In fact, we will give Tao's proof [30] that the maximal function conjectures are equivalent.

Chapter 2

Basic Analysis

We wish to study estimates of the form

$$||f_\delta^*||_{L^q(S^{n-1})} \leq \mathbf{K}_\delta(p,n)||f||_{L^p(\mathbb{R}^n)}, \tag{2.1}$$

for $1 \leq q \leq \infty$. As we have seen in Proposition 1.2.3 (2), we expect the bound in this estimate to take the form $\mathbf{K}_\delta(p,n) = C_{\epsilon,p,n}\delta^{-\alpha-\epsilon}$ for some fixed $\alpha \geq 0$ and an arbitrary $\epsilon > 0$. In this chapter we will acquire some basic tools for dealing with these types of estimates and gain some intuition into their subtlety and scope.

Section 2.1 will be a short list of some of the immediate simplifications we can make when proving an estimate as in (2.1). Section 2.2 will take us through an analysis of the local nature of the problem and reduce matters further to the case when $f \in L^p$ has support contained in the unit n-ball. In Section 2.3 we will show explicitly how estimates on the maximal function imply estimates on the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$ and so show that Conjecture 1.2.2 implies the two set conjectures of the previous chapter. Section 2.4 is less essential to the subsequent analysis, but it will serve as a bridge between our approach to the problem and the various approaches that can be found in much of the other literature. It will also acquaint the reader with some of the subtleties of working with off-diagonal, i.e. $L^p \to L^q$, type estimates.

2.1 Easy Observations

One immediate thing to notice about the estimate (2.1) is that it will always suffice to assume that f is nonnegative. Indeed, this follows from the definition of the maximal function itself and simply because $f \in L^p(\mathbb{R}^n)$ a priori. More precisely, we have the following.

Fact 2.1.1. *If* (2.1) *holds for nonnegative $f \in L^p$, then it holds for all functions in L^p.*

Proof. $||f||_p = ||(|f|)||_p$ for any $f \in L^p(\mathbb{R}^n)$, so using equation (1.3) or (1.4) we see that

$$||f_\delta^*||_q = ||(|f|)_\delta^*||_q \leq \mathbf{K}_\delta(p,n)||(|f|)||_p = \mathbf{K}_\delta(p,n)||f||_p;$$

thus if (2.1) holds for $f \geq 0$, then it holds for all $f \in L^p$. □

Fact 2.1.1 will be used repeatedly and without mention in the remainder of the essay.

Another easy observation we can make is that we can work with a simplified form of the Kakeya operator, one where we restrict the domain of the maximal function itself.

Fact 2.1.2. *In the context of* (2.1)*, the domain of the maximal function f_δ^*, initially defined on S^{n-1}, can be restricted to the set of all directions that lie within $\frac{1}{10}$ of the vertical e_n.*[1]

Verifying the fact will allow us to define and briefly work with several objects that will prove to be invaluable in our later analyses.

Definition 2.1.3. *We say a set $\Omega \subset S^{n-1}$ is* ***r-separated*** *if for all $\omega, \omega' \in \Omega$ we have $|\omega - \omega'| \geq r$. We also say that a set $\Omega \subset S^{n-1}$ is* ***maximally r-separated*** *if Ω is r-separated and any other set properly containing Ω is not.*

Proof of Fact 2.1.2. To prove the claim, we let Ω be a maximally $\frac{1}{10}$-separated set such that $e_n \in \Omega$. This separation defines a covering of S^{n-1} by caps C_i centered at each point of Ω of radius $\frac{1}{10}$. Since $|S^{n-1}| < \infty$, we know that the number of such caps is just some constant N depending on n. Let C_1 be the cap with center e_n. Now

$$\begin{aligned} ||f_\delta^*||_{L^q(S^{n-1})}^q &= \int_{S^{n-1}} f_\delta^*(e)^q de \\ &= \int_{\bigcup_{i=1}^N C_i} f_\delta^*(e)^q de \\ &\leq \sum_{i=1}^N \int_{C_i} f_\delta^*(e)^q de \\ &= \sum_{i=1}^N ||f_\delta^*||_{L^q(C_i)}^q, \end{aligned}$$

so (2.1) would follow by proving the estimate

$$||f_\delta^*||_{L^q(C_i)} \leq \mathbf{K}_\delta(p,n)||f||_p \tag{2.2}$$

[1]Note that there is nothing special about the subset of directions we have chosen here, as any spherical cap of fixed size (not dependent on δ) would suffice.

for each i. But we can always rotate any cap so that its center is at e_n; thus we should be able to reduce the matter to only verifying (2.2) for $i = 1$. Indeed, suppose (2.2) holds for $i = 1$. For any other i, let U be the rotation that maps the center of C_i to e_n. Then

$$||f_\delta^*||_{L^q(C_i)} = \left(\int_{C_i} f_\delta^*(e)^q de\right)^{1/q} = \left(\int_{C_1} f_\delta^*(U^{-1}e)^q de\right)^{1/q}. \tag{2.3}$$

Playing with the definition of the maximal function, we see that

$$\begin{aligned} f_\delta^*(U^{-1}e) &= \sup_{T \| U^{-1}e} \frac{1}{|T|} \int_T f(y)dy \\ &= \sup_{U(T) \| e} \frac{1}{|U(T)|} \int_{U(T)} f(U^{-1}y)dy \\ &= (f_U)_\delta^*(e) \end{aligned}$$

where $f_U(x) = f(U^{-1}x)$. Thus by hypothesis, (2.3) becomes

$$\left(\int_{C_1} (f_U)_\delta^*(e)^q de\right)^{1/q} \leq \mathbf{K}_\delta(p,n)||f_U||_p = \mathbf{K}_\delta(p,n)||f||_p,$$

verifying the claim. □

So it will suffice to work with this restricted version of the maximal function; in fact, for the remainder of this essay we will work solely with this restricted maximal function. This will help us avoid some annoying technicalities in many of our geometrical arguments by not having to consider how antipodal directions interact. In addition, this will also allow us to comfortably use the approximations $\sin\theta \sim \theta$ and $\cos\theta \sim 1 - \frac{\theta^2}{2}$ whenever θ denotes the angle between two directions within $\frac{1}{10}$ of the vertical. Indeed, we will use these approximations often and without mention in much of the following. From this point, we will take f_δ^* to denote this restricted maximal function.

A final simple observation we can make is the following.

Fact 2.1.4. *If $\mathbf{K}_\delta(p,n) = O(\delta^{-\alpha-\epsilon})$ in (2.1), then we must have that $p \geq \frac{n}{1+\alpha+\epsilon}$.*

Proof. To see this, let $f = \chi_{B(0,\delta)}$. Then for all $e \in S^{n-1}$, the tube $T_e^\delta(0)$ contains $B(0,\delta)$ and so $f_\delta^*(e) = \frac{|B(0,\delta)|}{|T_e^\delta(0)|} \gtrsim \delta$; hence, $||f_\delta^*||_p \sim \delta$. Now since $||f||_p \sim \delta^{\frac{n}{p}}$, we see that we require $\delta \lesssim \delta^{\frac{n}{p}-\alpha-\epsilon}$, or equivalently, $p \geq \frac{n}{1+\alpha+\epsilon}$. □

These types of results are the flavor of the present chapter. By continuing to study the fundamental nature of estimates like (2.1), we can become more comfortable with their implications, nuances and structure before we must dive into the main theorems of Chapters 4, 5 and 6.

2.2 Spatial Localization

By definition, f_δ^* lives on a compact space; when viewed as an operator, it takes functions living on all of $\mathbb{R}^n$ and maps them to functions living on S^{n-1}. It is the objective of this section to show that the behavior of the Kakeya operator is completely determined by how it acts on functions locally. This is essentially an artifact of f_δ^* being an averaging operator over precompact sets of fixed size. This is the first significant reduction to the Kakeya problem that we will make and it will be used in virtually all subsequent results.

Proposition 2.2.1. *Let $p \geq n$ and let $B(0,1)$ denote the unit n-ball centered at the origin. If (2.1) holds for all $f \in L^p$ with support contained in $B(0,1)$, then (2.1) holds for all $f \in L^p(\mathbb{R}^n)$ with $\mathbf{K}_\delta(p,n)$ replaced by $C_0 \cdot \mathbf{K}_\delta(p,n)$ for some absolute constant C_0.*

Perhaps the natural thing to hope for is to approximate an arbitrary L^p function by one with compact support and use a density argument to extend the estimate to all of L^p. The problem with this method is that it forces the bound $\mathbf{K}_\delta(p,n)$ in (2.1) to depend on the supports of our approximating C_0^∞ functions. Clearly this no good. Instead, we will decompose our arbitrary L^p function into pieces with compact and essentially disjoint support. This will allow us to take advantage of the fact that our maximal function takes an average over sets of fixed size, namely $\sim \delta^{n-1}$.

To simplify the proof of the proposition, we begin with a lemma that will exploit both the doubling property and the translation invariance of Lebesgue measure.

Lemma 2.2.2. *Let $p \geq n$, C some absolute constant. If (2.1) holds for all $f \in L^p$ with support contained in $B(0,1)$, then (2.1) holds for all $f \in L^p$ with supported contained in $B(a,C)$ for any $a \in \mathbb{R}^n$, again with $\mathbf{K}_\delta(p,n)$ replaced by $C_1 \cdot \mathbf{K}_\delta(p,n)$ for some absolute constant C_1.*

Proof. Let $f \in L^p$, $a \in \mathbb{R}^n$ be such that $\mathrm{supp}(f) \subset B(a,C)$. Write

$$f_\delta^*(e) = \sup_{T \| e} \frac{1}{|T|} \int_T f(y)dy$$

and make the substitution $x = \frac{y-a}{C}$ in the integral. If we define $f_U(x) = f(Cx+a)$, then we have

$$f_\delta^*(e) = \sup_{T \| e} \frac{C}{|T|} \int_{\frac{1}{C}(T-a)} f_U(x)dx. \tag{2.4}$$

Now $\frac{1}{C}(T-a)$ is a $(\frac{1}{C} \times \frac{\delta}{C} \times \cdots \times \frac{\delta}{C})$-tube centered near the origin. We can cover this tube by $T - \frac{a}{C}$ and thus (2.4) becomes

$$\begin{aligned} f^*_\delta(e) &\leq \sup_{T\|e} \frac{C}{|T|} \int_{T-\frac{a}{C}} f_U(x)dx \\ &= \sup_{T\|e} \frac{C}{|T|} \int_T f_U(x)dx \\ &\sim (f_U)^*_\delta(e) \end{aligned}$$

So using our hypothesis,

$$\begin{aligned} ||f^*_\delta||_p &\lesssim ||(f_U)^*_\delta||_p \\ &\leq \mathbf{K}_\delta(p,n)||f_U||_p \\ &\sim \mathbf{K}_\delta(p,n)||f||_p, \end{aligned}$$

verifying the lemma. □

Proof of Proposition 2.2.1. We start by covering $\mathbb{R}^n$ with closed integer boxes, i.e. sets of the form

$$Q_{\bar{m}} = \prod_{j=1}^n [m_j, m_j + 1]$$

with $\bar{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. This essentially partitions $\mathbb{R}^n$ so that we may take an arbitrary $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and define $f_{\bar{m}}(x) = f(x)\chi_{Q_{\bar{m}}}(x)$ so that

$$f(x) = \sum_{\bar{m}} f_{\bar{m}}(x) \quad a.e. \tag{2.5}$$

Of course, everything is with respect to Lebesgue measure on $\mathbb{R}^n$. Notice that equality holds everywhere in the interior of any $Q_{\bar{m}}$, but that the righthand side of (2.5) is a constant multiple of $f(x)$ depending only on the ambient dimension n whenever x is on the boundary of some integer box (and so in the intersection of a fixed number of integer boxes, again depending on the dimension of the space). If we think of both sides as merely functions in L^p, then there is no ambiguity in taking the equality at face-value. Indeed, this is how we will exploit it in the proof.

Fix $p \geq n$ and assume that (2.1) holds for any L^p function with support inside $B(0,1)$. Fix $e \in S^{n-1}$; then there exists some $T^\delta_e(a)$ such that

$$f^*_\delta(e) \leq \frac{2}{|T^\delta_e(a)|} \int_{T^\delta_e(a)} f(x)dx = \frac{2}{|T^\delta_e(a)|} \int_{T^\delta_e(a)} \sum_{\bar{m}} f_{\bar{m}}(x)dx. \tag{2.6}$$

Note that the first inequality in (2.6) is not a cheat; indeed, such a tube $T^\delta_e(a)$ must exist by definition of the supremum. Clearly there must also exist some $\bar{m}_0 \in \mathbb{Z}^n$ such that

$T_e^\delta(a) \cap Q_{\bar{m}_0} \neq \emptyset$. Since $T_e^\delta(a)$ has length 1, we see that $T_e^\delta(a) \cap Q_{\bar{m}} = \emptyset$ for all integer boxes not immediately adjacent to $Q_{\bar{m}_0}$. To be more precise, this means that

$$T_e^\delta(a) \cap Q_{\bar{m}} \neq \emptyset \ \forall \ \bar{m} \in \mathbb{Z}^n : Q_{\bar{m}} \cap Q_{\bar{m}_0} \neq \emptyset. \tag{2.7}$$

As in our discussion of equation (2.5), there are only finitely many multi-indeces $\bar{m}$ that satisfy this relation for any fixed $\bar{m}_0$ and this number is some determinable constant depending only on n. This allows us to localize the operator by only examining a finite number of the $f_{\bar{m}}$; consequently, we have that (2.6) is equal to

$$\frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} \sum\nolimits_{\bar{m}}^{*} f_{\bar{m}}(x)dx \leq \sum\nolimits_{\bar{m}}^{*} (f_{\bar{m}})_\delta^*(e), \tag{2.8}$$

where $\sum_{\bar{m}}^*$ denotes the sum over all indices $\bar{m}$ satisfying (2.7). We dominate (2.8) by applying Hölder and simplify:

$$\begin{aligned} \sum\nolimits_{\bar{m}}^{*} (f_{\bar{m}})_\delta^*(e) &\leq \left(\sum\nolimits_{\bar{m}}^{*} 1^{p'}\right)^{1/p'} \left(\sum\nolimits_{\bar{m}}^{*} |(f_{\bar{m}})_\delta^*(e)|^p\right)^{1/p} \\ &\lesssim_n ||(f_{\bar{m}})_\delta^*(e)||_{l^p(\mathbb{Z}^n)}, \end{aligned}$$

where p' is the dual exponent to p. Notice that the condition (2.7) has disappeared; indeed, we only required it to apply Hölder. Thus we arrive at our critical pointwise inequality:

$$f_\delta^*(e) \lesssim ||(f_{\bar{m}})_\delta^*(e)||_{l^p(\mathbb{Z}^n)}.$$

Taking L^p norms and raising both sides to the pth power, we see that

$$\begin{aligned} ||f_\delta^*||_{L^p(S^{n-1})}^p &\lesssim \int_{S^{n-1}} \sum_{\bar{m}} |(f_{\bar{m}})_\delta^*(e)|^p de \\ &= \sum_{\bar{m}} ||(f_{\bar{m}})_\delta^*||_{L^p(S^{n-1})}^p. \end{aligned} \tag{2.9}$$

We want to apply our hypothesis to the functions $f_{\bar{m}}$ but unfortunately they are not quite supported in the unit ball. For any fixed $\bar{m}$, this is easy to remedy though. Notice that $f_{\bar{m}}$ is supported in $Q_{\bar{m}}$ and that $Q_{\bar{m}} \subset B(a, C)$ for some $a \in \mathbb{R}^n$ and some absolute constant C. This constant is the scaling factor necessary to engulf a unit n-cube by an n-ball of this radius; evidently then, we can take $C > \sqrt{n}$ and apply Lemma 2.2.2. Thus,

$$\sum_{\bar{m}} ||(f_{\bar{m}})_\delta^*||_{L^p(S^{n-1})}^p \lesssim \sum_{\bar{m}} \mathbf{K}_\delta(p,n)^p ||f_{\bar{m}}||_{L^p(\mathbb{R}^n)}^p = \mathbf{K}_\delta(p,n)^p ||f||_{L^p(\mathbb{R}^n)}^p. \tag{2.10}$$

Combining this with (2.9) proves the proposition. □

We note that Proposition 2.2.1 can be strengthened to make a spatial locality statement about $L^p \to L^q$ estimates for $q > p$. The same proof that we have given will work with a minor adjustment to the final line. We cannot immediately write $\sum_{\bar{m}} ||f_{\bar{m}}||_p^q = ||f||_p^q$ in (2.10), so instead we apply the binomial theorem to bound the lefthand side by the righthand side.

We end this section by remarking that a statement completely analogous to Proposition 2.2.1 holds for the Nikodym maximal function. More precisely, we have

Proposition 2.2.3. *If*

$$||f_\delta^{**}||_{L^q(\mathbb{R}^n)} \leq \mathbf{N}_\delta(p,n)||f||_{L^p(\mathbb{R}^n)}, \tag{2.11}$$

holds for all $f \in L^p$ *with support contained in* $B(0,1)$*, then* (2.11) *holds for all* $f \in L^p(\mathbb{R}^n)$ *with* $\mathbf{N}_\delta(p,n)$ *replaced by* $C_0 \cdot \mathbf{N}_\delta(p,n)$ *for some absolute constant* C_0*. Here,* $\mathbf{N}_\delta(p,n)$ *is a constant depending only on* p *and* n *for any fixed* $\delta \ll 1$*.*

The proof is identical to that of Proposition 2.2.1, as the reader will have no trouble verifying. We will need to utilize both propositions in this section to prove our main result of Chapter 3, although Proposition 2.2.3 will not be needed beyond that point.

2.3 Maximal Function Estimates and Hausdorff Dimensionality

Suppose we are given an estimate on the maximal function of the form

$$\forall\, \epsilon > 0\ \exists\, C_\epsilon : ||f_\delta^*||_{L^q(S^{n-1})} \leq C_\epsilon \delta^{-\alpha-\epsilon}||f||_{L^p(\mathbb{R}^n)}, \tag{2.12}$$

for some $\alpha \geq 0$. Given such an estimate, we want to attain a lower bound on the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$ and thus demonstrate the relationship between Conjectures 1.1.7 and 1.2.2. Specifically, we will establish the following:

Proposition 2.3.1. *Let* K *be a Kakeya set in* $\mathbb{R}^n$*. Suppose* (2.12) *holds for some* $\alpha \geq 0$*,* $p, q < \infty$*. Then* $\dim_H(K) \geq n - p\alpha$*.*

Notice that if $\alpha = 0$, $q = p = n$, then (2.12) reduces to the Kakeya maximal function conjecture. This proposition is the essential component that ties the analytical problem to the geometric one. This is so useful that we will prove something even more general in order to afford ourselves a wider range of application. First though, we make a definition.

Definition 2.3.2. *An operator* T *is* ***restricted weak type (p,q)*** *with norm* $\leq A$*, written*

$$||Tf||_{q,\infty} \leq A||f||_{p,1},$$

if $|\{x : |T\chi_E(x)| \geq \lambda\}| \leq \left(\frac{A|E|^{\frac{1}{p}}}{\lambda}\right)^q$ *for all sets* E *with finite measure and all* $\lambda \in (0, 1]$*, where* χ_E *denotes the characteristic function of the set* E*.*

Any strong type estimate as in (2.12) implies the analogous restricted weak type estimate. This implication is not reversible in general, although we will see in Section 2.4 that for our problem, the two notions are often as good as equivalent. For an excellent exposition on the general theory of restricted weak type operators and their relation to strong type ones, see [2] or [28].

Proposition 2.3.3. *Let* K *be a Kakeya set in* $\mathbb{R}^n$ *and suppose*

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^*||_{q,\infty} \leq C_\epsilon \delta^{-\alpha-\epsilon}||f||_{p,1}, \tag{2.13}$$

for some $\alpha \geq 0$*,* $p, q < \infty$*. Then* $\dim_H(K) \geq n - p\alpha$*.*

We see that expression (2.13) is just the condition that the Kakeya operator is restricted weak type (p, q) with norm bounded by $C_\epsilon \delta^{-\alpha-\epsilon}$. Obviously then Proposition 2.3.3 is weaker than Proposition 2.3.1 and so more widely applicable; the former proposition will turn out to be very useful in Chapter 5.

Proof. We follow Wolff's presentation [32]. As such, we begin by making a key reduction. Suppose K is a Kakeya set in $\mathbb{R}^n$ and suppose that

$$\sum_j r_j^d \gtrsim 1,\ \forall\ \bigcup_{j=1}^{\infty} B(x_j, r_j) \supseteq K,\ r_j \leq 1. \tag{2.14}$$

Then it follows that for any $1 \geq \epsilon > 0$, $H_d^\epsilon(K) \gtrsim 1$ and so $H_d(K) \gtrsim 1$. Thus, $H_d(K) \neq 0$ and so $\dim_H(K) \geq d$. Therefore, to prove our proposition, it suffices to conclude that (2.14) holds for all $d < n - p\alpha$.

Fix $d < n - p\alpha$. For each $e \in S^{n-1}$ we know that K contains a unit line segment parallel to e; denote this line segment by I_e and let a_e denote its midpoint. Fix a covering of K by balls $B(x_j, r_j)$ with $r_j \leq 1$. Define for each $l \in \mathbb{Z}^+$,

$$J_l = \{j : 2^{-l} \leq r_j \leq 2^{-(l-1)}\},$$

$$K_l = K \cap \left(\bigcup_{j \in J_l} B(x_j, r_j)\right),$$

and

$$S_l = \left\{e \in S^{n-1} : |I_e \cap K_l| \geq \frac{c}{l^2}\right\},$$

where $c = \frac{6}{\pi^2}$. In this context, $|A|$ denotes the one-dimensional Lebesgue measure of the set. The sets J_l partition the balls in our covering according to their size, which is comparable to their radii. If we fix and l and consider the characteristic function of the balls corresponding to J_l (call it g), then since we are assuming the Kakeya operator is restricted weak type with norm as in (2.13), we know that

$$\left|\left\{e \in S^{n-1} : g^*_{2^{-l}}(e) \geq \lambda\right\}\right|^{1/q} \lesssim \lambda^{-1} 2^{-l(\frac{n}{p} - \alpha - \epsilon}|J_l|^{1/p}.$$

Thus, if we can bound the size of the set on the lefthand side by an appropriate factor of λ then we are done; indeed, we immediately have that

$$\sum_j r_j^d \geq \sum_l |J_l| \cdot 2^{-l(n-p\alpha-p\epsilon)},$$

so we see that we require this factor of λ to not shrink too fast as l increases. What this really means is that we want to make sure the set $\{e : g^*_{2^{-l}}(e) \geq \lambda_l\}$ does not shrink in size too fast. Notice that the set on the lefthand side is essentially a union of sets of the form S_l.

We note that for all $e \in S^{n-1}$ there exists some l depending on e such that $|I_e \cap K_l| \geq \frac{c}{l^2}$. To prove this, suppose the contrary. Then for all l,

$$1 = |I_e| = |\bigcup_{l=1}^{\infty} (I_e \cap K_l)\,| \leq \sum_{l=1}^{\infty} |I_e \cap K_l| \lneqq c \sum_l \frac{1}{l^2} = 1,$$

a contradiction; thus, such an l must exist and so we have the inclusion $S^{n-1} \subseteq \bigcup S_l$. Since the reverse inclusion is trivial, we see that in fact $S^{n-1} = \bigcup S_l$. This is a key point as it allows us to decompose our set of directions into ones where we have some quantitative control over the intersection of our Kakeya set with its covering.

Now by the pigeonhole principle, there must exist some $l \in \mathbb{Z}^+$ such that $|S_l| \gtrsim \frac{1}{l^2}$. We can see this more explicitly by again supposing the contrary. Then

$$\sum_l \sum_{e \in S_l} |I_e \cap K_l| \leq \sum_l |S_l| \cdot |I_e| \lesssim \sum_l \frac{1}{l^2} \lesssim 1;$$

however, since K is Kakeya and $\bigcup K_l = K$ by definition, we must have that the sum on the lefthand side is infinite, contradicting the calculation.

So fix an l such that $|S_l| \gtrsim \frac{1}{l^2}$ and define

$$F_l = \bigcup_{j \in J_l} B(x_j, 2r_j).$$

Note that

$$\forall\, y \in K_l,\ F_l \supseteq B(y, 2^{-l}). \tag{2.15}$$

To see that this is true, pick any $y \in K_l$; then $y \in B(x_j, r_j)$ for some $j \in J_l$. Thus, $y \in B(x_j, 2r_j)$. Now let $z \in B(y, 2^{-l})$; then

$$|x_j - z| \leq |x_j - y| + |y - z| < r_j + 2^{-l} \leq 2r_j.$$

So $B(y, 2^{-l}) \subseteq B(x_j, 2r_j)$ and (2.15) follows.

All this work leads to a critical bound; if $e \in S_l$, then

$$|T_e^{2^{-l}}(a_e) \cap F_l| \gtrsim \frac{1}{l^2}|T_e^{2^{-l}}(a_e)|. \tag{2.16}$$

To prove this, we note that $T_e^{2^{-l}}(a_e) \cap F_l \supseteq T_e^{2^{-l}}(a_e) \cap B(y, 2^{-l})$ for all $y \in K_l$ by (2.15), so in particular this is true for all $y \in I_e \cap K_l$. Thus,

$$T_e^{2^{-l}}(a_e) \cap F_l \supseteq T_e^{2^{-l}}(a_e) \cap \left(\bigcup_{y \in I_e \cap K_l} B(y, 2^{-l}) \right).$$

The righthand side above is the portion of the 2^{-l}-tube with I_e as its principal axis that "shadows" $I_e \cap K_l$; i.e. this is only the portion(s) of the tube with $I_e \cap K_l$ as principal axis. The measure of this set is then bounded below by $\frac{c}{l^2}|T_e^{2^{-l}}(a_e)|$.

With this in place, we define $f = \chi_{F_l}$ and calculate

$$\left| \left\{ e \in S^{n-1} : f^*_{2^{-l}}(e) \gtrsim \frac{1}{l^2} \right\} \right| \geq \left| \left\{ e \in S^{n-1} : \frac{|F_l \cap T_e^{2^{-l}}(a_e)|}{|T_e^{2^{-l}}(a_e)|} \gtrsim \frac{1}{l^2} \right\} \right|$$

which is bounded below by $|S_l| \gtrsim \frac{1}{l^2}$ by (2.16). Thus,

$$\left| \left\{ e \in S^{n-1} : f^*_{2^{-l}}(e) \gtrsim \frac{1}{l^2} \right\} \right| \gtrsim \frac{1}{l^2}. \tag{2.17}$$

Now we use our assumption that equation (2.13) holds. Fix $\epsilon > 0$ and note that (2.13) gives

$$\begin{aligned} \left| \left\{ e \in S^{n-1} : f^*_{2^{-l}}(e) \gtrsim \frac{1}{l^2} \right\} \right|^{1/q} &\leq C_\epsilon \cdot l^2 2^{l(\alpha+\epsilon)} \cdot |F_l|^{1/p} \\ &\lesssim C_\epsilon \cdot l^2 2^{l(\alpha+\epsilon)} (2^{-ln} \cdot |J_l|)^{1/p} \\ &= C_\epsilon \cdot l^2 2^{-l(\frac{n}{p}-\alpha-\epsilon)} |J_l|^{1/p}. \end{aligned} \tag{2.18}$$

Combining equations (2.17) and (2.18) then gives us

$$l^{-2/q} \lesssim C_\epsilon \cdot l^2 2^{-l(\frac{n}{p}-\alpha-\epsilon)} |J_l|^{1/p},$$

or equivalently

$$|J_l| \gtrsim l^{-2p(1+\frac{1}{q})} 2^{l(n-p\alpha-p\epsilon)}. \tag{2.19}$$

Therefore,

$$\sum_j r_j^{n-p\alpha-p\epsilon} \geq \sum_l |J_l| \cdot 2^{-l(n-p\alpha-p\epsilon)} \gtrsim \sum_l l^{-2p(1+\frac{1}{q})} \gtrsim 1.$$

For any $d < n - p\alpha$, we may find an $\epsilon > 0$ such that $d = n - p\alpha - p\epsilon$ and so by (2.14) we have that $\dim_H(K) \geq n - p\alpha$, completing the proof of the proposition.

□

Our proposition required that (2.13) hold for some $\alpha \geq 0$, $p, q < \infty$, $p \geq n$. Notice that if we let $\alpha = \frac{n}{p} - 1$, then our proposition says that $\dim_H(K) \geq p$. Also note that if $\alpha = 0$, then $\dim_H(K) = n$.

One other crucial thing to notice is that Proposition 2.3.3 tells us that $L^p \to L^p$ estimates such as (2.1), i.e. estimates on the diagonal, do indeed suffice. This is evident from the lower bound given by the proposition which does not depend on the value of q whatsoever. Hence, if we can prove an $L^p \to L^p$ estimate, then this is just as good as an $L^p \to L^q$ estimate in terms of all three Kakeya conjectures.

2.4 Interpolation, the Critical Line and the Diagonal

This section is largely motivational. We cannot resolve the Kakeya conjectures in full, but we can make partial progress. In the following chapters, we will prove certain bounds on the Kakeya operator acting on the diagonal and so improve what we know. We will see here that when looking for partial Kakeya bounds, there is a natural form to expect. The main purpose of this section will be to motivate this natural form and to convince the reader that it is always sufficient to work with diagonal estimates when wanting to make progress on any of the three Kakeya conjectures. This will require various methods of interpolation, although things are not quite as straight-forward as we might hope for simply because the operators in question are never linear. Interpolation is usually cited as the reason why we expect certain intermediate Kakeya bounds to be true, but details are rarely given. Here, we bridge this gap.

Proposition 2.4.1. *If the Kakeya maximal function conjecture (Conjecture 1.2.2) holds, then for every $1 \leq p \leq n$, $1 \leq q \leq (n-1)\frac{p}{p-1}$, we have the estimate:*

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^*||_q \leq C_\epsilon \delta^{-\frac{n}{p}+1-\epsilon} ||f||_p. \tag{2.20}$$

Of course we see that if $p = n$ in the proposition above, then what we have is the full maximal function conjecture; but notice also that if $p = 1$, then this estimate is already

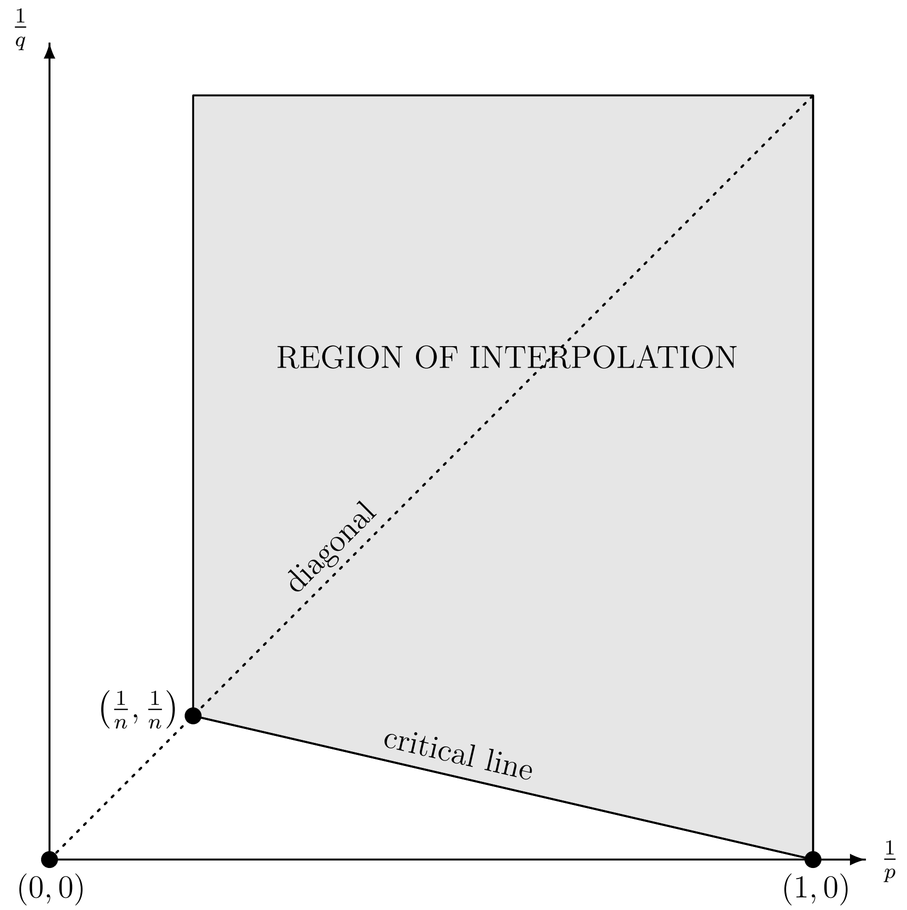

Figure 2.1: *The region of interpolation corresponding to the full Kakeya conjecture.*

known to us by the $L^1 \to L^\infty$ estimate of Proposition 1.2.3. This suggests that (2.20) should follow from interpolation with the trivial $L^1 \to L^\infty$ bound and the conjectured $L^n \to L^n$ bound. Hence, we can make partial progress on the Kakeya problem by proving an estimate in this hypothesized intermediate range. Tightening p as close to n as we can will give us a stronger estimate and so move us closer to the full conjecture. It will be convenient to think of the proposed range of p and q where the operator obeys the estimate (2.20) pictorially, as depicted above. Note that we refer to the line parameterized by $1 \leq p \leq n$, $q = (n-1)\frac{p}{p-1}$ as the critical line.

This all seems reasonable, but we have to make sure that (2.20) really does follow by interpolation as proposed. After all, if it did not, then the proposed bound in (2.20) is seemingly arbitrary and we would have no reason to suspect that it is true, much less to exert any energy into proving such a partial estimate.

It is very tempting to use the prototypical interpolation result of Riesz-Thorin. This would give us the precise range of p we are after with $q = (n-1)\frac{p}{p-1}$ and a bound with a convex combination of the exponents in the $L^1 \to L^\infty$ and $L^n \to L^n$ bounds. A calculation shows that this would give the bound in (2.20).

Unfortunately, Riesz-Thorin does not apply since the Kakeya operator is *sublinear*, not linear. Consequently, to make our interpolation argument, we will have to take a more delicate approach.

We will make use of several classical notions and results in interpolation theory. We will not dwell on the full scope of these matters in detail, referring the reader to the bibliography for a proper treatment of the general theory. We will however provide proofs of two necessary results since we have not been able to find a good resource for their derivations. For organizational purposes, we list these definitions and results now (with references) before proceeding to the proof of Proposition 2.4.1.

Definition 2.4.2. *An operator T is of* ***weak type (p,q)*** *with norm $\leq A$ if*

$$|\{x : |Tf(x)| \geq t\}| \leq \left(\frac{A||f||_p}{t}\right)^q$$

for all $f \in L^p$ and all $t \in (0,1]$. Also, we call $\lambda_f(t) = |\{x : |Tf(x)| \geq t\}|$ the ***distribution function*** *of f at t with respect to the operator T.*

Note that this definition is the same as Definition 2.3.2, except that for an operator to be of restricted weak type, the distributional inequality above need only hold for characteristic functions of sets with finite measure rather than for all functions in L^p.

Theorem 2.4.3 (Marcinkiewicz Interpolation, Theorem 4.13 and Corollary 4.14 in [2])**.** *Suppose $1 \leq p_0 < p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$ with $p_i \leq q_i$ $(i = 0, 1)$ and $q_0 \neq q_1$. Let $0 < \theta < 1$ and define p and q by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.21}$$

If T is a nonnegative, sublinear operator of weak types (p_0, q_0) and (p_1, q_1), with respective norms A_0 and A_1, then T is of strong type (p, q) with norm $\lesssim \max\{A_0, A_1\}$ where the implicit constant depends only on p_0, q_0, p_1, q_1 and θ.

We need a final pair of propositions specific to operators enjoying the spatial locality property of Proposition 2.2.1. Of course for our purposes, there is no harm in restricting attention to the Kakeya operator alone. These results are not as classical, so we will provide proofs. The second proposition is stated in more generality than is necessary to give a proof of Proposition 2.4.1, but we will need this strength of generality for another result.

Proposition 2.4.4. *If the Kakeya operator is of strong type (p, q) with norm $\leq A$, then it is also of strong type (p, r) with norm $\leq A$ for all $1 \leq r < q$.*

Proof. Pick $1 \leq r < q$ and write

$$\begin{aligned} ||f^*_\delta||_r &= \left(\int f^*_\delta(e)^r de \right)^{1/r} \\ &= ||(f^*_\delta)^r \chi_{\text{supp} f^*_\delta}||_1^{1/r}. \end{aligned}$$

Since we know, by Proposition 2.2.1, that it is sufficient to consider only f with support in the unit ball, we know that we can take $\text{supp} f^*_\delta \subseteq B(0, 2)$. Hence, since $r \leq q$, by Hölder we have

$$||f^*_\delta||_r \lesssim ||(f^*_\delta)^r||_{q/r}^{1/r} = ||f^*_\delta||_q \leq A||f||_p$$

by hypothesis. □

For the next proposition, we make a definition.

Definition 2.4.5. *An operator T is of* ***restricted strong type (p,q)*** *with norm $\leq A$ if it is of strong type (p, q) with norm $\leq A$ for all characteristic functions of finite measure sets, i.e. if*

$$||T\chi_E||_q \leq A||\chi_E||_p = A|E|^{1/p}$$

for all sets E with $|E| < \infty$.

Proposition 2.4.6. *With our parameters defined as in Theorem 2.4.3, suppose in addition that $q_0 < q_1$. If the Kakeya operator is of restricted weak types (p_0, q_0) and (p_1, q_1), with respective norms $\leq A_0$ and $\leq A_1$, then it is also of restricted strong type (p, q) with norm $\lesssim A_0^{1-\theta} A_1^\theta$.*

Notice that this proposition will give us the Riesz-Thorin type bound on the interpolated norms in Proposition 2.4.1.

Proof. Using the distribution function, our hypothesis that the Kakeya operator is of restricted weak types (p_0, q_0) and (p_1, q_1) with respective norms $\leq A_0$ and $\leq A_1$ becomes

$$\lambda(t) \leq \left(\frac{A_0|E|^{1/p_0}}{t} \right)^{q_0} \quad \text{and} \quad \lambda(t) \leq \left(\frac{A_1|E|^{1/p_1}}{t} \right)^{q_1}, \tag{2.22}$$

for all subsets $E \subseteq B(0, 1)$ and all $t \in (0, 1]$. Here, we have abbreviated our notation and written $\lambda(t) = \lambda_{\chi_E}(t)$.

We require a simple identity for distribution functions, namely,

$$||f||_p^p = p \int_0^\infty t^{p-1} \lambda_f(t) dt. \tag{2.23}$$

This identity is immediate for characteristic functions f, since then we have

$$p\int_0^\infty t^{p-1}\lambda_{\chi_E}(t)dt = p\int_0^1 t^{p-1}|E|dt = |E| = ||\chi_E||_p^p;$$

so we can extend to the general case by approximating any $f \in L^p$ from below by simple functions[2].

Returning to (2.22), we may assume that the first bound on $\lambda(t)$ is less than or equal to the second bound. Solving the corresponding inequality for t, we see that this requires

$$t \leq \left(A_0^{-q_0}A_1^{q_1}|E|^{\frac{q_1}{p_1}-\frac{q_0}{p_0}}\right)^{\frac{1}{q_1-q_0}} = A,$$

where we denote the bound by A for simplicity. We want to show that the Kakeya operator is of restricted strong type (p,q) for any p and q given by (2.21). Fix one such pair and use the distributional identity (2.23) to estimate

$$\begin{aligned} ||(\chi_E)_\delta^*||_q^q &= q\int_0^A t^{q-1}\lambda(t)dt + q\int_A^\infty t^{q-1}\lambda(t)dt \\ &\leq q\int_0^A t^{q-q_0-1}A_0^{q_0}|E|^{q_0/p_0}dt + q\int_A^\infty t^{q-q_1-1}A_1^{q_1}|E|^{q_1/p_1}dt \\ &\lesssim A^{q-q_0}A_0^{q_0}|E|^{q_0/p_0} + A^{q-q_1}A_1^{q_1}|E|^{q_1/p_1}, \end{aligned} \tag{2.24}$$

since $q_0 < q < q_1$ by hypothesis. Now we plug in for A and simplify. We will only work through the details for the first term in (2.24) since the second term follows by the identical algebra.

Notice that by definition, we have

$$1-\theta = \frac{\frac{1}{q_1}-\frac{1}{q}}{\frac{1}{q_1}-\frac{1}{q_0}} = \frac{q_0-\frac{q_1q_0}{q}}{q_0-q_1} = \frac{q_0}{q}\left(\frac{q-q_1}{q_0-q_1}\right),$$

and so we also have

$$\theta = \frac{q_1}{q}\left(\frac{q_0-q}{q_0-q_1}\right).$$

Plugging into the first term of (2.24) for A and simplifying, we find

$$\begin{aligned} A^{q-q_0}A_0^{q_0}|E|^{q_0/p_0} &= A_0^{q_0\left(\frac{q_1-q}{q_1-q_0}\right)}A_1^{q_1\left(\frac{q-q_0}{q_1-q_0}\right)}|E|^{\frac{q_0}{p_0}\left(\frac{q_1-q}{q_1-q_0}\right)+\frac{q_1}{p_1}\left(\frac{q-q_0}{q_1-q_0}\right)} \\ &= A_0^{(1-\theta)q}A_1^{\theta q}|E|^{q\left(\frac{\theta}{p_1}+\frac{1-\theta}{p_0}\right)} \\ &= A_0^{(1-\theta)q}A_1^{\theta q}|E|^{q/p}. \end{aligned}$$

[2]See [28] for a thorough investigation of the distribution function and its many applications.

It now immediately follows that

$$||(\chi_E)^*_\delta||_q \lesssim A_0^{1-\theta} A_1^\theta ||\chi_E||_p,$$

proving the proposition. □

We are now ready to prove the main motivational result of the section.

Proof of Proposition 2.4.1. As we remarked after the statement of the proposition, the classic interpolation result of Riesz-Thorin is useless for our purposes. Now, we may be tempted to just apply Marcinkiewicz's theorem and interpolate between (n, n) and $(1, \infty)$, but this will not give the desired convex combination of the exponents in the (n, n) and $(1, \infty)$ bounds. Instead, we note that any strong type estimate is automatically of restricted weak type and apply Proposition 2.4.6 to interpolate between the two endpoints. This gives us the desired bound $\lesssim \delta^{-\frac{n}{p}+1-\epsilon}$ along the critical line, but it is only of restricted strong type. To improve this to a strong bound, we use the following procedure.

Notice that the Kakeya operator is strong type $(1, q)$ with norm $\lesssim \delta^{-(n-1)}$ for any $1 \leq q \leq \infty$. We use Proposition 2.4.6 again to interpolate between (n, n) and $(1, q)$ for all q and so obtain our restricted strong type bound $\lesssim \delta^{-\frac{n}{p}+1-\epsilon}$ everywhere in the triangular region between the critical line and the diagonal (see Figure **??**). We are going to use the $L^\infty \to L^\infty$ estimate of Proposition 1.2.3 to improve our bound along the critical line to be of strong type.

Choose any pair (p, q) on the critical line (so $q = (n-1)\frac{p}{p-1}$) and let $\theta > 0$ be small. We want to interpolate using Marcinkiewicz between (∞, ∞) and some (p_0, q_0) in our triangular region where we have the restricted strong type estimates so that the point (p, q) lies on the line of interpolation between the two. Considering our defining relations (2.21), we see that we should require

$$p_0 = (1-\theta)p \quad \text{and} \quad q_0 = (1-\theta)q.$$

Applying Marcinkiewicz interpolation, we find that the strong (p, q) norm of the Kakeya operator is bounded by

$$\max\{1, \delta^{-\frac{n}{p_0}+1-\epsilon}\} = \delta^{-\frac{n}{p_0}+1-\epsilon} = \delta^{-\frac{n}{(1-\theta)p}+1-\epsilon}. \tag{2.25}$$

Now as we let $\theta \to 0$, the form of the above bound does not change and in fact we see that if $\theta < \frac{\epsilon}{\frac{n}{p}+\epsilon}$ then the bound in (2.25) is dominated by

$$\delta^{-\frac{n}{p}+1-2\epsilon} = \delta^{-\frac{n}{p}+1-\epsilon'},$$

and so we have the appropriate strong type (p, q) bound for this point (p, q) on the critical line. Clearly, we can repeat this exact procedure for any other (p, q) on the critical line and so extend to strong type estimates everywhere along this line. Finally, a quick application of Proposition 2.4.4 everywhere along this line allows us to deduce the necessary strong type estimates everywhere within our proposed region, $1 \le p \le n$, $1 \le q \le (n-1)\frac{p}{p-1}$. □

It is often more convenient to prove a restricted weak type estimate than to prove a strong one; indeed, this is precisely what Bourgain and Wolff have done to prove their Kakeya estimates (Chapters 5 and 6). We show now why this type of estimate suffices.

Proposition 2.4.7. *Suppose the Kakeya operator has restricted weak type-*(p_0, q_0) *norm* $\le A$ *for some* $1 \le p_0 \le n$, $1 < q_0 \le (n-1)\frac{p}{p-1}$. *Then for all* $\epsilon > 0$, *the Kakeya operator has strong type-*(p, q) *norm* $\lesssim A\delta^{-\epsilon}$ *for all* $1 \le p \le p_0$, $1 \le q \le q_0$.

The situation is illustrated below.

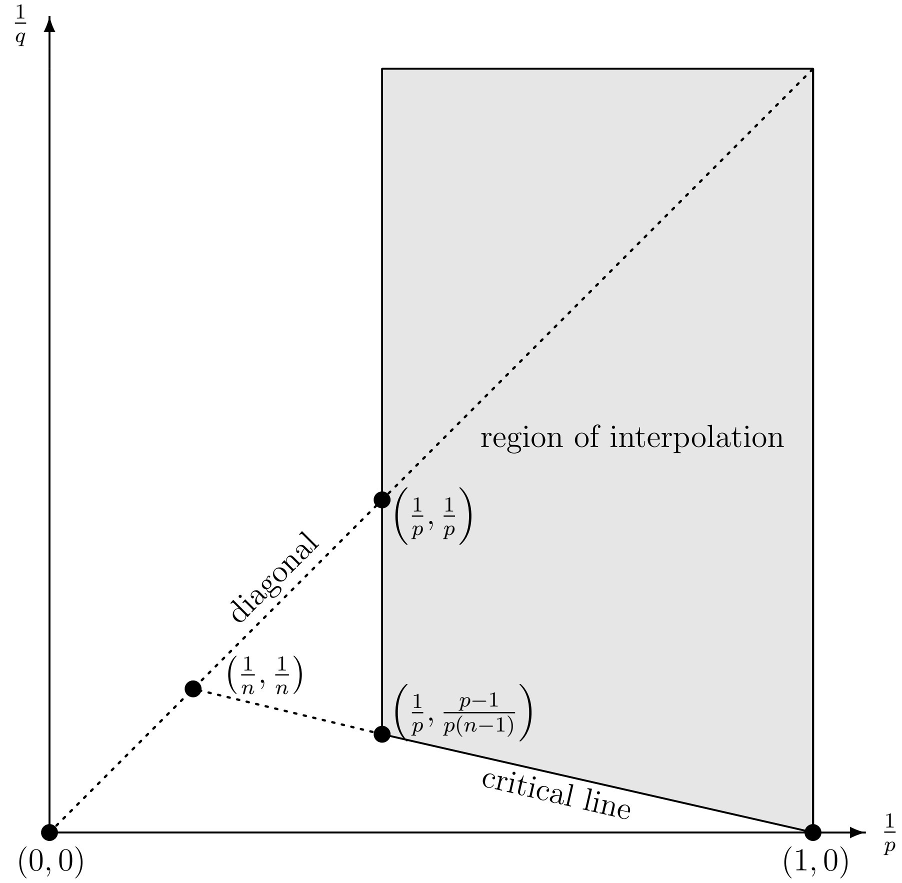

Figure 2.2: *The region of interpolation corresponding to a partial Kakeya estimate of strong type* $\left(p, (n-1)\frac{p}{p-1}\right)$.

Proof. The proof is almost identical to the previous. We start by interpolating between (p_0, q_0) and $(1, q)$ for all $1 \leq q \leq \infty$ using Proposition 2.4.6 to obtain restricted strong type bounds $\leq A$ everywhere in the triangular region defined by the vertices (p_0, q_0), $(1, \infty)$ and $(1, 1)$ (note though that we do not yet have anything more than the restricted weak type estimate at the single vertex (p_0, q_0)). Now for every pair (p, q) on the lower edge of this triangle, we improve our estimate to a strong type bound $\lesssim A\delta^{-\epsilon}$ using Marcinkiewicz to interpolate between an appropriate (p', q') to the northeast of the pair (p, q) and the origin, (∞, ∞). Finally, we apply Hölder (Proposition 2.4.4) to extend our strong type bounds everywhere in the region $1 \leq p \leq p_0$, $1 \leq q \leq q_0$. □

In Proposition 2.3.1 we showed, among other things, that in the context of the Kakeya set conjectures, it is sufficient to prove Kakeya estimates on the diagonal. We showed in Proposition 2.4.1 that we can expect to make partial progress on the Kakeya maximal functions conjecture by proving a certain type of bound along the critical line. The real meaning behind Proposition 2.4.7 then is that, to make progress on the maximal function conjecture, it is sufficient to prove the appropriate Kakeya estimates on the diagonal. This simplifies matters slightly, allowing us to only consider the L^p behavior of the Kakeya operator as it acts on functions in L^p.

Definition 2.4.8. *We say that $\mathcal{K}(\boldsymbol{p})$ holds if we have the Kakeya estimate*

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^*||_{L^p(S^{n-1})} \leq C_\epsilon \cdot \delta^{-\frac{n}{p}+1-\epsilon}||f||_{L^p(\mathbb{R}^n)}$$

for all f in L^p. We say that $\mathcal{N}(\boldsymbol{p})$ holds if we have the Nikodym estimate

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^{**}||_{L^p(\mathbb{R}^n)} \leq C_\epsilon \cdot \delta^{-\frac{n}{p}+1-\epsilon}||f||_{L^p(\mathbb{R}^n)}$$

for all f in L^p.

This notation allows us to quickly identify different Kakeya (and Nikodym) estimates and we will use it often in the subsequent chapters.

Chapter 3

Equivalence of the Kakeya and Nikodym Maximal Function Conjectures

This chapter can be skipped if desired by the reader, but we felt it a good idea to include it here in order to motivate a stronger desire for Kakeya estimates. In no way though do the following chapters depend on the material of this one. However, many of the analytical tools and techniques that we will need later will be applied in this chapter as well, so it may serve the reader well to study the proofs in this chapter in anticipation of the following.

This chapter will consist of a proof of Tao's theorem that the Kakeya maximal function conjecture and the Nikodym maximal function conjecture are equivalent. Furthermore, we will show that any Hausdorff dimension bound for Kakeya sets in $\mathbb{R}^n$ serves as an identical Hausdorff dimension bound for Nikodym sets in $\mathbb{R}^n$. Recalling the notation of Definition 2.4.8, we have the following theorem.

Theorem 3.0.1 (Tao). *We have $\mathcal{K}(p)$ if and only if we have $\mathcal{N}(p)$.*

The proof is lengthy and naturally splits into two parts which we will divide into separate sections.

3.1 Nikodym Implies Kakeya

That $\mathcal{N}(p)$ implies $\mathcal{K}(p)$ is easier to see and so we prove this first. The implication will follow from a combination of a geometrical fact and a pointwise inequality. These facts will allow us to pass between the Kakeya and Nikodym maximal functions on an annulus of constant size, which will be sufficient to establish the main result. The geometrical fact is a containment lemma, relating a tube with a given direction to a tube containing a given point; we exploit this containment then to set up a pointwise estimate, a simple inequality relating the two maximal functions directly. We state these facts as lemmas and prove them now.

Lemma 3.1.1. *Let C be some absolute constant to be determined and let $|x| \sim 1$. If T is any δ-tube oriented in the $\frac{x}{|x|}$ direction, then there exists some $(C\delta^{-1} \times C\delta \times \cdots \times C\delta)$-tube T' containing the point $\frac{x}{\delta}$ such that $T \cap B(0,1) \subset T'$.*

Proof. Now, we want to show the estimate on an annulus, $|x| \sim 1$. The key point here is that $|x|$ is bounded away from zero and infinity, so there is no harm in assuming $\frac{1}{2} < |x| < 2$. Fix some x in this annulus and let ϕ_1 be the rotation that sends $\frac{x}{|x|}$ to $e_n = (0, 0, \ldots, 0, 1)$, the nth canonical basis vector for $\mathbb{R}^n$.

Pick any δ-tube T oriented in the $\frac{x}{|x|}$ direction whose intersection with $B(0,1)$ is nonempty. Then $\phi_1(T) \cap B(0,1)$ is at most the restriction to the unit ball of a cylinder oriented in the e_n direction with e_j-thickness 2δ for all $1 \leq j \leq n-1$. Denote this cylinderical strip by $S(\underline{r}, 0)$, where $\underline{r} = (r_1, \ldots, r_{n-1})$ denotes the point on the principal axis of our cylinder (i.e. the principal axis of our tube $\phi_1(T)$) that intersects the hyperplane $\{r_n = 0\}$. If we can find a C such that there exists a $T' \ni \frac{\phi_1(x)}{\delta}$ that contains this cylindrical strip, then we are done. The relevant geometry here is that the further this strip is from the origin, the larger the angle needed to orient T' with respect to the e_n-axis, and as this angle becomes larger, this will require C to be larger.

Denote the center of the tube $\phi_1(T)$ by a and note that $a \in B(0, \frac{3}{2})$, else $\phi_1(T)$ could not intersect $B(0,1)$. In fact, since $\phi_1(T) \parallel e_n$, we see that if we write $a = (\underline{a}, a_n)$, then we must have $\underline{a} \in B^{n-1}(0, 1+\delta)$, the $(n-1)$-ball centered at the origin with radius $1+\delta$. From the last paragraph, we note the formal inclusion:

$$\phi_1(T) \cap B(0,1) \subseteq S(\underline{a}, 0) = \{|z| \leq 1 : \underline{z} - \underline{a} \in B^{n-1}(0, \delta)\}. \tag{3.1}$$

We want take advantage of the rotational invariance of the problem again to simplify the analysis. Accordingly, we let ϕ_2 be the rotation that *fixes* e_n while sending $\frac{a}{|a|}$ to $\frac{a'}{|a'|}$, where $a' = (\sqrt{|a|^2 + a_n^2}, 0, \ldots, 0, a_n)$. Then ϕ_2 rotates our tube $\phi_1(T)$ so that its principal axis intersects the e_1-axis while remaining parallel to e_n. With these simplifications in place, we can now define the $(C\delta^{-1} \times C\delta \times \cdots \times C\delta)$-tube T' that will contain our rotated δ-tube $\phi_2 \circ \phi_1(T)$.

We choose the natural orientation e for T' defined by the line segment connecting $\phi_2 \circ \phi_1(\frac{x}{\delta}) = \phi_1(\frac{x}{\delta})$ and $(\underline{a'}, 0)$; so since $|\phi_1(x)| = |x|$, we set

$$e = \frac{(a'_1, 0, \ldots, 0, -\frac{|x|}{\delta})}{|(a'_1, 0, \ldots, 0, -\frac{|x|}{\delta})|},$$

where $a_1' = \sqrt{|a|^2 + a_n^2}$. We want to choose the center of T' to be close to $\frac{x}{\delta}$; therefore, define the center to be the point $b = (\underline{b}, b_n)$ where $\underline{b} \in B^{n-1}(0, \delta)$ and $0 < b_n \leq \frac{|x|}{\delta} \sim \frac{1}{\delta}$. All we need now is the constant C to ensure that $\phi_2(S(\underline{a}, 0)) = S(\underline{a'}, 0) \subset T'$.

We remind the reader that by definition we have

$$T' = \{y \in \mathbb{R}^n : |(y - b) \cdot e| \leq \frac{C}{\delta},\ |\text{proj}_{e^\perp}(x - a)| \leq \delta\}, \tag{3.2}$$

where $e^\perp$ is the hyperplane through the origin with e as normal vector. There are many different bases $\{\zeta_1, \ldots, \zeta_{n-1}\}$ we can choose for the subspace $e^\perp$, but it will be computationally necessary to pick a particularly convenient one here. Clearly it is possible to choose some ζ_1 contained in $\pi(e_1, e_n)$, the 2-plane generated by the vectors e_1 and e_n, since e is in this plane. Accomplishing this, we see that we can then choose $\zeta_2 = e_2, \ldots, \zeta_{n-1} = e_{n-1}$. Thus we immediately have that if $z \in S(\underline{a'}, 0)$, then

$$|(z - b) \cdot \zeta_j| = |(z - b) \cdot e_j| \leq |z_j| + |b_j| \lesssim \delta,$$

for all $2 \leq j \leq n - 1$. So we only need to concern ourselves with the containment in the 2-plane $\pi(e_1, e_n)$. Our goal now is to show that ζ_1 is almost exactly e_1, as to be expected from our defining parameters.

Let θ denote the angle in the 2-plane $\pi(e_1, e_n)$ between e_1 and ζ_1. Referring to the figure below, we can use similar triangles to note that this angle is the same as the angle between e and e_n.

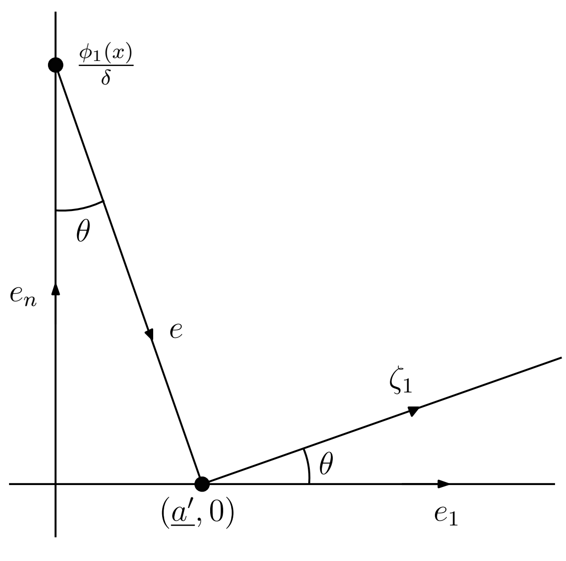

Figure 3.1: The almost orthogonality of e and e_1, and of e_n and ζ_1.

Thus we have

$$\sin^2\theta = \frac{(a_1')^2}{(a_1')^2 + (\delta^{-1}|\phi_1(x)|)^2} \lesssim \delta^2$$

since $|a_1'| \lesssim 1$ and $|\phi_1(x)| = |x| \sim 1$. Consequently, $\theta = |e_1 - \zeta_1| \lesssim \delta$. Looking back at (3.2) and noting that by construction we require $a' \in T'$, we can now assert that if $z \in S(\underline{a'}, 0)$, then

$$\begin{aligned} |(z-b)\cdot\zeta_1| &= |(z-a')\cdot\zeta_1 + (a'-b)\cdot\zeta_1| \\ &\lesssim |(z-a')\cdot(\zeta_1 - e_1) + (z-a')\cdot e_1| + \delta \\ &\lesssim |z-a'|\delta + |z_1 - a_1'| + \delta \\ &\lesssim \delta, \end{aligned}$$

by the definition of $S(\underline{a'}, 0)$ in (3.1).

Finally, we check that if $z \in S(\underline{a'}, 0)$, then

$$\begin{aligned} |(z-b)\cdot e| &= \frac{1}{|(a_1', 0, \ldots, 0, -\frac{|x|}{\delta})|}\left|(z_1 - b_1)a_1' - (z_n - b_n)\frac{|x|}{\delta}\right| \\ &\lesssim \frac{1 + \frac{1}{\delta^2}}{\frac{1}{\delta}} \\ &\lesssim \frac{1}{\delta}, \end{aligned}$$

as was to be shown. Hence, there exists some constant C dependent only on the ambient dimension such that $S(\underline{a'}, 0) \subset T'$.

Unraveling our transformations, we see that for any given δ-tube T intersecting the unit ball, oriented in the $\frac{x}{|x|}$ direction for some given $|x| \sim 1$, we may find a $(C\delta^{-1} \times C\delta \times \cdots \times C\delta)$-tube T' such that

$$T \cap B(0,1) \subset \phi_1^{-1} \circ \phi_2^{-1}(T'),$$

with $\frac{x}{\delta} \in \phi_1^{-1} \circ \phi_2^{-1}(T')$ as required. Observing that $\phi_1^{-1} \circ \phi_2^{-1}(T')$ is still a $(C\delta^{-1} \times C\delta \times \cdots \times C\delta)$-tube, we conclude the lemma. □

Lemma 3.1.1 will allow us to establish the critical pointwise estimate necessary for the proof of the first half of Tao's theorem. We remark that this kind of containment lemma often arises when attempting to acquire L^p estimates on the Kakeya maximal function or the Nikodym maximal function. This is an artifact of the definition of the maximal operator as a supremum over various *integrals*. Naturally then, to obtain an inequality for the L^p behavior of the operator, the most obvious thing to try is to bound the integrand by a simpler object while possibly enlarging the domain of integration. This enlargement of the domain of integration usually arises from a containment lemma of the previous sort.

Lemma 3.1.2. *Let C be some absolute constant and define $f_\delta(x) = f(\frac{x}{\delta})$. Then for all $|x| \sim 1$ and any $f \in L^p$ with support contained in $B(0,1)$, we have the pointwise estimate*

$$f_\delta^*\left(\frac{x}{|x|}\right) \lesssim \delta^{-1}(f_\delta)^{**}_{C\delta^2}(x). \tag{3.3}$$

Proof. We begin with some simplifications. As mentioned in the previous chapter, we will always take $f \in L^p(\mathbb{R}^n)$ to be nonnegative with support contained in the unit n-ball when working with Kakeya estimates. As we have seen, this is sufficient to conclude any Kakeya estimate on all of L^p by Fact 2.1.1 and Proposition 2.2.1. It is easy to see that it suffices to assume the same when working with the analogous Nikodym estimates. Indeed, we can utilize Proposition 2.2.3 and observe that nonnegativity follows in the identical fashion as for Kakeya. Our proof of Tao's theorem will require these assumptions on both the Kakeya and Nikodym operators in order to simplify the problem.

Fix $|x| \sim 1$. Using the definition of the Kakeya maximal function, we rewrite the lefthand side of (3.3) as

$$f_\delta^*\left(\frac{x}{|x|}\right) \sim \sup_{T || \frac{x}{|x|}} \frac{1}{\delta^{n-1}} \int_T f(y)dy. \tag{3.4}$$

Considering the righthand side of (3.3), we use the definition of the Nikodym maximal function to write

$$\delta^{-1}(f_\delta)^{**}_{C\delta^2}(x) \sim \sup_{\widetilde{T} \ni x} \frac{1}{\delta^{2n-1}} \int_{\widetilde{T}} f\left(\frac{y}{\delta}\right) dy,$$

where $\widetilde{T}$ is a $(C \times C\delta^2 \times \cdots \times C\delta^2)$-tube. Substituting $z = \frac{y}{\delta}$, we see that this reduces to

$$\sup_{T' \ni \frac{x}{\delta}} \frac{1}{\delta^{n-1}} \int_{T'} f(z)dz. \tag{3.5}$$

where T' is a $(C\delta^{-1} \times C\delta \times \cdots \times C\delta)$-tube.

Comparing (3.4) and (3.5), we see that in order to prove our pointwise estimate it suffices to show that for all $T || \frac{x}{|x|}$ there exists a $T' \ni \frac{x}{\delta}$ such that

$$\int_T f(y)dy \lesssim \int_{T'} f(y)dy. \tag{3.6}$$

But this is precisely the statement of Lemma 3.1.1; hence, (3.6) holds. The pointwise estimate (3.3) follows accordingly. □

Proof of Theorem 3.0.1, reverse implication. With the pointwise estimate in place for all $|x| \sim 1$, we can now show that $\mathcal{N}(p)$ implies $\mathcal{K}(p)$. Taking the L^p-norm of the lefthand side of equation (3.3) over the annulus $|x| \sim 1$, we find that

$$\left|\left|f_\delta^*\left(\frac{x}{|x|}\right)\right|\right|_{L^p(|x|\sim 1)}^p \sim \int_{S^{n-1}} \int_{r\sim 1} f_\delta^*(e)^p dr de \sim ||f_\delta^*||_{L^p(S^{n-1})}^p,$$

where we have converted to spherical coordinates and then integrated out the radial measure. Applying the pointwise estimate, we have

$$||f_\delta^*||_{L^p(S^{n-1})} \lesssim \delta^{-1}||(f_\delta)_{C\delta^2}^{**}||_{L^p(|x|\sim 1)} \lesssim \delta^{-1}||(f_\delta)_{C\delta^2}^{**}||_{L^p(\mathbb{R}^n)}.$$

By hypothesis, $\mathcal{N}(p)$ holds so the above is dominated by

$$\lesssim \delta^{-1}(\delta^2)^{-\frac{n}{p}+1-\epsilon}||f_\delta||_{L^p(\mathbb{R}^n)}.$$

Noting that $||f_\delta||_p = \delta^{\frac{n}{p}}||f||_p$ and performing the obvious simplification in the exponents, we arrive at the estimate

$$||f_\delta^*||_{L^p(S^{n-1})} \lesssim \delta^{-\frac{n}{p}+1-2\epsilon}||f||_{L^p(\mathbb{R}^n)},$$

which is the desired estimate $\mathcal{K}(p)$. It is worthwhile to point out the extra loss of $\delta^{-\epsilon}$ that is absorbed into the estimate $\mathcal{K}(p)$. As we have seen, we can always absorb this kind of increase on the bound without lossing any quantitative power in the context of Hausdorff dimension bounds. This completes the proof of the first direction of Tao's theorem. □

3.2 Kakeya Implies Nikodym

We show the former implication, that $\mathcal{K}(p)$ implies $\mathcal{N}(p)$. Just as in the previous section, the proof will be divided into several lemmas.

We begin by making a few reductions. Since f is supported on $B(0,1)$, we must have that f_δ^{**} is supported on $B(0,2)$ - this is simply because any tube that is taken in the supremum must intersect the support of f and every tube has length 1. So the operator is compactly supported (in a sense). We will use this fact to apply a method of Carbery [9].

Instead of working over $\mathbb{R}^n$, we want to "freeze" the x_n variable and consider a kind of Kakeya (and Nikodym) estimate over $\mathbb{R}^{n-1}$, i.e. in the first $(n-1)$ variables. More precisely, we have the following lemma.

Lemma 3.2.1. *Consider a Nikodym estimate frozen in the x_n variable of the form*

$$||f_\delta^{**}(\underline{x}, x_n)||_{L^p(\mathbb{R}^{n-1})} \lesssim \delta^{-\frac{n}{p}+1-\epsilon}||f||_{L^p(\mathbb{R}^n)}. \tag{3.7}$$

If (3.7) *holds for $x_n = 0$, then $\mathcal{N}(p)$ holds.*

Proof. Suppose that (3.7) holds for all $|x_n| \leq 2$. Then we notice that

$$\begin{aligned}
||f_\delta^{**}||_{L^p(\mathbb{R}^n)}^p &= \int_{B(0,2)} f_\delta^{**}(x)^p dx \\
&= \int_{-2}^{2} \int_{B^{n-1}(0,2)} f_\delta^{**}(\underline{x}, x_n)^p d\underline{x} dx_n \\
&= \int_{-2}^{2} ||f_\delta^{**}(\underline{x}, x_n)||_{L^p(\mathbb{R}^{n-1})}^p dx_n \\
&\lesssim \int_{-2}^{2} \delta^{-n+p-\epsilon p} ||f||_p^p dx_n \\
&\sim \delta^{-n+p-\epsilon p} ||f||_p^p.
\end{aligned}$$

Thus, if we have the frozen estimate (3.7) at all $|x_n| \leq 2$, then we have the full Nikodym estimate $\mathcal{N}(p)$. We want to do better though and show that it actually suffices to prove (3.7) for the single frozen value $x_n = 0$. This is possible by the spatial locality of the Nikodym problem and by the translation invariance of Lebesgue measure.

Suppose that $||f_\delta^{**}(\underline{x}, 0)||_p \lesssim \delta^{-\frac{n}{p}+1-\epsilon} ||f||_p$ holds. Then for any $|a| \leq 2$,

$$\begin{aligned}
||f_\delta^{**}(\underline{x}, a)||_p^p &= \int \left(\sup_{T \ni (\underline{x}, a)} \frac{1}{|T|} \int_T f(y) dy \right)^p d\underline{x} \\
&= \int \left(\sup_{T \ni (\underline{x}, a)} \frac{1}{|T|} \int_{T-(\underline{0}, a)} f(z + (\underline{0}, a)) dz \right)^p d\underline{x},
\end{aligned} \tag{3.8}$$

where $z = y - (\underline{0}, a)$. Let $f_a(z) = f(z + (\underline{0}, a))$ so that f_a is supported on $B(0, 3)$. Notice that (3.8) then becomes

$$\begin{aligned}
\int \left(\sup_{T \ni (\underline{x}, 0)} \frac{1}{|T|} \int_T f_a(z) dz \right)^p d\underline{x} &= ||(f_a)_\delta^{**}(\underline{x}, 0)||_p^p \\
&\lesssim \delta^{-n+p-\epsilon p} ||f_a||_p^p \\
&= \delta^{-n+p-\epsilon p} ||f||_p^p,
\end{aligned}$$

verifying the lemma. □

So it suffices to show the frozen estimate at $x_n = 0$; this will simplify the following geometry considerably. Notice though that just because we freeze $x_n = 0$ for the maximal function, does not mean that we are freezing anything in the support of f; indeed, any tube containing the point $(\underline{x}, 0)$ for $\underline{x} \in B^{n-1}(0, 2)$ must lie within 1 of the x_n-axis and so can still intersect any portion of the support of f. Clearly, the modulus of any x_n

in our support is bounded from above, but we would also like to bound its range from below and so utilize a similar reasoning as in Section 3.1 when we analyzed points on the annulus $|x| \sim 1$. To accomplish this, we first show that it suffices to assume f is supported on the intersection of $B(0,1)$ with the slab $0 < x_n \leq 1$.

Lemma 3.2.2. *Suppose $\mathcal{N}(p)$ holds for all f with support in $B(0,1) \cap \{0 < x_n \leq 1\}$. Then $\mathcal{N}(p)$ holds for all $f \in L^p$.*

Proof. We already know by Proposition 2.2.3 that to prove $\mathcal{N}(p)$, it suffices to consider only those $f \in L^p$ with support in $B(0,1)$. Accordingly, we pick any such f and define

$$f_+(x) = f(x)\chi_{(0,1]}(x_n),$$

$$f_-(x) = f(\underline{x}, -x_n)\chi_{(0,1]}(x_n);$$

then we have f_+ and f_- both supported on $0 < x_n \leq 1$ with $f = f_+ + f_-$ almost everywhere. Now $f_\delta^{**}(x) \leq (f_+)_\delta^{**}(x) + (f_-)_\delta^{**}(x)$ by sublinearity and so

$$||f_\delta^{**}||_p^p \leq ||(f_+)_\delta^{**} + (f_-)_\delta^{**}||_p^p \leq (||(f_+)_\delta^{**}||_p + ||(f_-)_\delta^{**}||_p)^p \tag{3.9}$$

by the triangle inequality. Applying Hölder, we see that

$$\begin{aligned}(||(f_+)_\delta^{**}||_p + ||(f_-)_\delta^{**}||_p)^p &\leq \left[(||(f_+)_\delta^{**}||_p^p + ||(f_-)_\delta^{**}||_p^p)^{1/p} \cdot (1^{p'} + 1^{p'})^{1/p'}\right]^p \\ &= 2^{p/p'}(||(f_+)_\delta^{**}||_p^p + ||(f_-)_\delta^{**}||_p^p).\end{aligned}$$

This inequality coupled with (3.9) then gives

$$\begin{aligned}||f_\delta^{**}||_p^p &\lesssim ||(f_+)_\delta^{**}||_p^p + ||(f_-)_\delta^{**}||_p^p \\ &\lesssim \delta^{-n+p-\epsilon p}(||f_+||_p^p + ||f_-||_p^p) \\ &= \delta^{-n+p-\epsilon p}||f||_p^p,\end{aligned}$$

since f_+ and f_- are both supported on $0 < x_n \leq 1$ and by hypothesis, our estimate holds for such functions. □

With that reduction in place, we now want to decompose the slab $\{0 < x_n \leq 1\}$ dyadically and use a scaling argument to consider only the largest term. This will allow us to restrict our attention to pointwise estimates over an annulur domain as in the proof of the first part of Tao's theorem. Subsequently, we have another lemma.

Lemma 3.2.3. *Suppose* (3.7) *holds with $x_n = 0$ for all f with support in $B(0,1) \cap \{\frac{1}{2} < x_n \leq 1\}$. Then $\mathcal{N}(p)$ holds for all $f \in L^p$.*

Proof. In light of Lemma 3.2.2, we see that we may assume the function f is supported on $B(0,1) \cap \{0 < x_n \leq 1\}$. Partition $(0,1]$ as $\bigcup_{k\geq 0}(2^{-k-1}, 2^{-k}]$ and define

$$f_k(x) = f(x)\chi_{(2^{-k-1},2^{-k}]}(x_n)$$

so that $f = \sum_{k\geq 1} f_k$. Notice

$$\int f_k(x)dx = \int_{x_n \in (\frac{1}{2},1]} f(\underline{x}, 2^{-k}x_n)2^{-k}dx$$

for any $k \geq 0$; thus, if we define $f_k'(\underline{x}, x_n) = f(\underline{x}, 2^{-k}x_n)\chi_{(\frac{1}{2},1]}(x_n)$, then f_k' is supported on the slab $\frac{1}{2} < x_n \leq 1$ for all $k \geq 0$ and $\int f_k = 2^{-k}\int f_k'$. Now by the triangle inequality and hypothesis, we have

$$\begin{aligned} ||f_\delta^{**}(\underline{x}, 0)||_p &\leq \sum_k ||(f_k)_\delta^{**}(\underline{x}, 0)||_p \\ &= \sum_k 2^{-k}||(f_k')_\delta^{**}(\underline{x}, 0)||_p \\ &\leq \delta^{-\frac{n}{p}+1-\epsilon}\sum_k 2^{-k}||f_k'||_p \\ &= \delta^{-\frac{n}{p}+1-\epsilon}\sum_k 2^{-k(1-\frac{1}{p})}||f_k||_p. \end{aligned} \tag{3.10}$$

Apply Hölder to the sum to bound (3.10) by

$$\begin{aligned} \delta^{-\frac{n}{p}+1-\epsilon}\left(\sum_k 2^{-k}\right)^{\frac{1}{p'}}\left(\sum_k ||f_k||_p^p\right)^{\frac{1}{p}} &= \delta^{-\frac{n}{p}+1-\epsilon}2^{\frac{1}{p'}}||f||_p \\ &\lesssim \delta^{-\frac{n}{p}+1-\epsilon}||f||_p. \end{aligned}$$

Applying Lemma 3.2.1, we conclude that we may indeed take f to be supported on the intersection $B(0,1) \cap \{\frac{1}{2} < x_n \leq 1\}$. □

Now we want to derive a pointwise estimate just as in Section 3.1 to get to the second part of Tao's theorem. Again, this is going to rely on some critical geometry in the form of a containment lemma. Define the transformation $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi(\underline{x}, x_n) = \left(\frac{\underline{x}}{x_n}, \frac{1}{x_n}\right).$$

Notice that $\phi^{-1} = \phi$, so that we may analyze either and glean the same information. This transformation will allow us to pass freely between our tubes of interest for the Kakeya operator (i.e. tubes with a fixed direction) and our tubes of interest for the Nikodym operator (i.e. tubes containing a fixed point). This statement takes a particularly clean form in $\mathbb{R}^2$: fix a direction $e \in S^1$ and pick any tube T_e^δ such that $T_e^\delta \cap \{\frac{1}{2} < x_2 \leq 1\} \neq \emptyset$ and $(x, 0) \in T_e^\delta$ for some arbitrary $|x| \leq 2$. Then the principal axis of this tube has slope $\frac{e_2}{e_1}$ and x_2-intercept equal to x, where $e = (e_1, e_2)$. Applying the transformation ϕ to this tube, we find that the transformed object is now essentially a $C\delta$-tube with principal axis having slope $\frac{1}{x}$ and x_2-intercept $\frac{e_1}{e_2}$. What we have accomplished then is to take a tube with fixed direction and variable x_2-intercept and transform it into a tube with variable direction and fixed x_2-intercept.

We want to make this analytically precise in $\mathbb{R}^n$ for any $n \geq 2$; thus, we have the following lemma.

Lemma 3.2.4. *Let $\underline{x} \in B^{n-1}(0, 2)$ be arbitrary and fix a direction $e \in S^{n-1}$. Also, let T_e^δ be a δ-tube such that $(\underline{x}, 0) \in T_e^\delta$. If ϕ is defined as before, then there exists some constant C, independent of $\underline{x}$, δ and e, such that*

$$\phi\left(T_e^\delta \cap \{1/2 < x_n \leq 1\}\right) \subset C \cdot T^\delta_{\frac{(\underline{x},1)}{|(\underline{x},1)|}}.$$

Proof. Clearly we may assume that $T_e^\delta \cap \{1/2 < x_n \leq 1\} \neq \emptyset$. Let $y \in T_e^\delta \cap \{1/2 < x_n \leq 1\}$. Then since $y \in T_e^\delta$ and $(\underline{x}, 0) \in T_e^\delta$, we know that y is contained in some line segment parallel to e that intersects the $\{x_n = 0\}$-hyperplane in a δ-neighborhood of $(\underline{x}, 0)$. Explicitly, we have a parameterization of y as

$$y = (\underline{x} + \underline{a}, 0) + t_y(\underline{e}, e_n) \tag{3.11}$$

for some $\underline{a} \in B^{n-1}(0, \delta)$, where $e = (\underline{e}, e_n)$ and $-1 \lesssim t_y \lesssim 1$. Note that we can immediately assume $t_y \geq 0$ by replacing t_y with $-t_y$ if necessary in the above parameterization. Now we also have that $y \in \{1/2 < x_n \leq 1\}$ and so this places a stronger restriction on the value of t_y. Indeed, since $1/2 < y_n \leq 1$, we see that

$$\frac{1}{2e_n} < t_y \leq \frac{1}{e_n}. \tag{3.12}$$

We can do better though. Since $T_e^\delta \cap \{1/2 < x_n \leq 1\} \neq \emptyset$, we see that e cannot be too far off the vertical $(0, \ldots, 0, 1)$; in fact, we have $|e_n| \geq y_n > 1/2$. We combine this with (3.12) to deduce that

$$\frac{1}{2} < t_y < 2, \tag{3.13}$$

since $|e_n| \leq 1$ trivially. So we have $t_y \sim 1$ and (3.11) is now in concord with our intuition.

Now we are in a suitable position to analyze $\phi(y)$. Define

$$r_{\phi(y)} = \frac{|(\underline{x}, 1)|}{t_y e_n}.$$

Now $\underline{x} \in B^{n-1}(0,2)$, so $1 \leq |(\underline{x},1)| \leq 3$. We combine this observation with (3.12) to notice that $1 \leq r_{\phi(y)} < 6$, and so $r_{\phi(y)} \sim 1$. Using (3.11) we have

$$\begin{aligned}\phi(y) &= \frac{1}{t_y e_n}(\underline{x} + \underline{a} + t_y \underline{e}, 1) \\ &= \left(\frac{\underline{e}}{e_n}, 0\right) + \frac{1}{t_y e_n}(\underline{a}, 0) + \frac{1}{t_y e_n}(\underline{x}, 1) \\ &= \left(\frac{\underline{e}}{e_n}, 0\right) + r_{\phi(y)}\frac{(\underline{a}, 0)}{|(\underline{x}, 1)|} + r_{\phi(y)}\frac{(\underline{x}, 1)}{|(\underline{x}, 1)|}.\end{aligned}$$

We claim that this point is contained in the tube $C \cdot T^{\delta}_{\frac{(\underline{x},1)}{|(\underline{x},1)|}}(\frac{\underline{e}}{e_n}, 0)$, for some C independent of the parameters $\underline{x}$, δ and e. To verify this, we need only appeal to the definition of a tube.

Calculate

$$\begin{aligned}\left|\left[\phi(y) - \left(\frac{\underline{e}}{e_n}, 0\right)\right] \cdot \frac{(\underline{x}, 1)}{|(\underline{x}, 1)|}\right| &= \frac{r_{\phi(y)}}{|(\underline{x}, 1)|^2}|(\underline{x} + \underline{a}, 1) \cdot (\underline{x}, 1)| \\ &\leq r_{\phi(y)}|(\underline{x} + \underline{a}, 1)| \cdot |(\underline{x}, 1)| \\ &< C_1,\end{aligned}$$

since $|a_j| \leq \delta < 1$ for all $1 \leq j \leq n-1$. Now let $v \in S^{n-1}$ be orthogonal to $\frac{(\underline{x},1)}{|(\underline{x},1)|}$; then we have

$$\begin{aligned}\left|\left[\phi(y) - \left(\frac{\underline{e}}{e_n}, 0\right)\right] \cdot v\right| &= \frac{r_{\phi(y)}}{|(\underline{x}, 1)|}|(\underline{a}, 0) \cdot v| \\ &\leq r_{\phi(y)} \sum_{j=1}^{n-1} |a_j||v_j| \\ &\leq r_{\phi(y)}(n-1)\delta \\ &\leq C_2\delta.\end{aligned}$$

Thus, letting $C = \max\{C_1, C_2\}$, we have $\phi(y) \in C \cdot T^{\delta}_{\frac{(\underline{x},1)}{|(\underline{x},1)|}}(\frac{\underline{e}}{e_n}, 0)$ as required. □

With the containment lemma in place, we are now ready to derive our key pointwise estimate.

Lemma 3.2.5. *For any f with support in $B(0,1) \cap \{1/2 < x_n \leq 1\}$, we have*

$$f^{**}_{\delta}(\underline{x}, 0) \lesssim (f \circ \phi_C)^*_{\delta}\left(\frac{(\underline{x}, 1)}{|(\underline{x}, 1)|}\right), \tag{3.14}$$

where $\phi(\underline{x}, x_n) = \left(\frac{\underline{x}}{x_n}, \frac{1}{x_n}\right)$ as before and $\phi_C(x) = \phi(2C \cdot x)$.

Proof. Fix an $\underline{x} \in B^{n-1}(0,2)$; then for any f with support in $B(0,1) \cap \{1/2 < x_n \leq 1\}$ there exists some δ-tube $T \ni (\underline{x}, 0)$ such that

$$f_\delta^{**}(\underline{x}, 0) \leq \frac{2}{|T|} \int_T f(y)dy$$
$$= \frac{2}{|T|} \int_{\phi^{-1}(T)} f(\phi(z))|D\phi(z)|dz.$$

Now $D\phi(z) = \frac{1}{z_n^{n+1}}$, so $1 \leq |D\phi(z)| < 2^{n+1}$. Recalling that $\phi^{-1} = \phi$, we have

$$f_\delta^{**}(\underline{x}, 0) \leq \frac{2^{n+2}}{|T|} \int_{\phi(T)} f(\phi(z))dz.$$

Now we apply Lemma 3.2.4 to obtain the pointwise estimate

$$f_\delta^{**}(\underline{x}, 0) \leq \frac{2^{n+2}}{|T|} \int_{C \cdot T^\delta_{\frac{(\underline{x},1)}{|(\underline{x},1)|}}} f(\phi(z))dz$$
$$\lesssim \frac{1}{|T|} \int_{T^\delta_{\frac{(\underline{x},1)}{|(\underline{x},1)|}}} f(\phi_C(z'))dz'$$
$$\lesssim (f \circ \phi_C)_\delta^* \left(\frac{(\underline{x}, 1)}{|(\underline{x}, 1)|} \right),$$

where we made the substitution $z = 2C \cdot z'$ in the middle inequality. □

We require a final fact about surface integrals before we can collect our results and finish the proof of Tao's theorem.

Fact 3.2.6. *Fix $k > 0$. Let $F : S^{n-1} \to \mathbb{R}$ be in $L^1(S^{n-1})$. Then*

$$\int_{B^{n-1}(0,k)} F\left(\frac{(x, 1)}{|(x, 1)|} \right) dx \lesssim \int_{S^{n-1}} F(e)de. \tag{3.15}$$

Proof. Let $y_i = \frac{x_i}{|(x,1)|}$ for all $1 \leq i \leq n-1$. Notice that $\frac{1}{|(x,1)|^2} = 1 - |y|^2$ and so $x_i = \frac{y_i}{\sqrt{1-|y|^2}}$. For any $1 \leq i \leq n-1$, we have

$$\frac{\partial x_i}{\partial y_i} = \frac{1}{\sqrt{1 - |y|^2}} + \frac{y_i^2}{(1 - |y|^2)^{3/2}} = \frac{1 - |y|^2 + y_i^2}{(1 - |y|^2)^{3/2}},$$

and for any $1 \leq i, j \leq n-1$ with $i \neq j$, we have

$$\frac{\partial x_i}{\partial y_j} = \frac{y_i y_j}{(1 - |y|^2)^{3/2}}.$$

Thus the Jacobian J of our transformation is bounded as follows:

$$|J| \lesssim \frac{\prod_i |1-|y|^2+y_j^2| + |\text{terms with a factor of } y_i \text{ for some } i|}{(1-|y|^2)^{\frac{3(n-1)}{2}}} \lesssim \frac{1}{(1-|y|^2)^{\frac{3(n-1)}{2}}}$$

since $|y| \lesssim \frac{k}{\sqrt{1+k^2}} \lesssim 1$. Therefore,

$$\int_{B^{n-1}(0,k)} F\left(\frac{(x,1)}{|(x,1)|}\right) dx \lesssim \int_{B^{n-1}\left(0,\frac{k}{\sqrt{1+k^2}}\right)} F(y,\sqrt{1-|y|^2}) \frac{dy}{(1-|y|^2)^{\frac{3(n-1)}{2}}}. \tag{3.16}$$

Now, we easily have that

$$\int_{S^{n-1}} F(e)de = \int_{|y|\leq 1} F(y,\sqrt{1-|y|^2}) \frac{dy}{\sqrt{1-|y|^2}} \tag{3.17}$$

by parameterizing S^{n-1} in the usual way. Notice that $(1-|y|^2)^{-\frac{3(n-1)}{2}}$ is comparable to $(1-|y|^2)^{-\frac{1}{2}}$ over the domain $B^{n-1}\left(0,\frac{k}{\sqrt{1+k^2}}\right)$ since $1-|y|^2 \geq \frac{1}{1+k^2}$. Thus, (3.16) is bounded by

$$\lesssim \int_{|y|\leq 1} F(y,\sqrt{1-|y|^2}) \frac{dy}{\sqrt{1-|y|^2}}.$$

Combining this with (3.17), we have inequality (3.15). □

With all the lemmas established, the remainder of the proof of Tao's theorem is an easy exercise.

Proof of Theorem 3.0.1, forward implication. Starting with Lemma 3.2.5, we take $L^p(B^{n-1}(0,2))$ norms of both sides of (3.14) to conclude that

$$\begin{aligned} ||f_\delta^{**}(\underline{x},0)||_{L^p(B^{n-1}(0,2))} &\lesssim ||(f\circ\phi_C)_\delta^*\left(\frac{(\underline{x},1)}{|(\underline{x},1)|}\right)||_{L^p(B^{n-1}(0,2))} \\ &\lesssim ||(f\circ\phi_C)_\delta^*||_{L^p(S^{n-1})}, \end{aligned}$$

where we have applied Fact 3.2.6 to establish the second inequality. Now $\mathcal{K}(p)$ holds by hypothesis, so

$$||(f\circ\phi_C)_\delta^*||_{L^p(S^{n-1})} \lesssim \delta^{-\frac{n}{p}+1-\epsilon}||f\circ\phi_C||_{L^p(\mathbb{R}^n)}.$$

Observing that $||f\circ\phi_C||_p \sim 2^{\frac{n+1}{p}}||f||_p$, we arrive at

$$||f_\delta^{**}(\underline{x},0)||_p \lesssim \delta^{-\frac{n}{p}+1-\epsilon}||f||_p,$$

verifying the frozen estimate at $x_n = 0$. Thus, by Lemma 3.2.1, we see that $\mathcal{N}(p)$ must hold. □

Although the proof of Tao's theorem required some work, we are now rewarded with the knowledge that any Kakeya estimate implies an analogous Nikodym estimate and vice-versa. This will become particularly relevant in the following three chapters where we derive our main three Kakeya estimates. In light of our work here, this will provide us with essentially identical Nikodym estimates for free. Perhaps what is better, we will establish lower bounds on the Hausdorff dimension of Kakeya and Nikodym sets in $\mathbb{R}^n$ simultaneously by simply appealing to Theorem 3.0.1

Chapter 4

L^2 Estimates

The Kakeya operator in two dimensions is well understood. This is greatly due to the fact that the ambient geometry in the plane is often transparent and always pictorially tractable. The Kakeya conjecture in $\mathbb{R}^2$ also enjoys a uniquely affable position in no small part because of the powerful duality properties of the L^2 Hilbert space. These niceties quickly break down when we elevate the dimension, but the methods and estimates in this chapter are highly classical and motivate the underlying techniques and further results in the remainder of this essay. Much of the now standard analysis has its roots in the original paper of Córdoba [11].

In his 1977 paper, Córdoba was concerned primarily with studying the Bochner-Riesz spherical summation operators[1] and their boundedness on $L^p(\mathbb{R}^n)$ for a certain range of p. Building largely off Fefferman's work [15], Córdoba analyzed the boundedness of the Nikodym maximal function on $\mathbb{R}^2$ to obtain a quantitative estimate on the norm of certain Bochner-Riesz operators. In our context, we have the following theorem.

Theorem 4.0.1 (Córdoba)**.** *In $\mathbb{R}^2$, we have the estimate:*

$$||f^*_\delta||_{L^2(S^1)} \lesssim \sqrt{\log \frac{1}{\delta}} \cdot ||f||_{L^2(\mathbb{R}^2)}.$$

4.1 Resolution of the Kakeya Conjecture in $\mathbb{R}^2$

Although Córdoba was concerned with matters other than the Kakeya problem, his result not only gives a quantitative upper bound for the norm of the maximal operator, it in fact gives the optimal one. This settles the Kakeya conjecture in two dimensions. Unfortunately, the proof offers little insight into the higher dimensional cases as it relies heavily upon the L^2 structure. The crucial geometrical lemma below will be our most

[1] See Chapter 7, section 3 for a definition and brief discussion.

valuable item in this chapter and will be utilized repeatedly to improve our understanding of the more complicated higher dimensional cases. Although Córdoba's estimate is on L^2, we state and prove the lemma in $\mathbb{R}^n$, as these critical geometric facts will be needed later in this greater generality.

Lemma 4.1.1 (Córdoba)**.** *For any pair of directions $e_k, e_l \in S^{n-1}$ and any pair of points $a, b \in \mathbb{R}^n$, we have the estimates*

$$\operatorname{diam}(T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b)) \lesssim \frac{\delta}{|e_k - e_l|}, \tag{4.1}$$

and

$$|T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b)| \lesssim \frac{\delta^n}{|e_k - e_l|}. \tag{4.2}$$

This lemma is the first critical geometric fact we need about intersections of tubes in $\mathbb{R}^n$ and we will use it repeatedly throughout the exposition. In line with our intuition, note that if the tubes lie very close together in space and in orientation (i.e. if $|e_k - e_l| \sim \delta$), then their intersection is nearly full, $|T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b)| \lesssim \delta^{n-1}$. If on the other hand the tubes are close in space but nearly orthogonal in orientation (i.e. if $|e_k - e_l| \sim 1$), then their intersection is essentially a δ-box in $\mathbb{R}^n$, $|T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b)| \lesssim \delta^n$. We proceed to the proof.

Proof. Clearly we may assume that the tubes lie near each other in space otherwise there is nothing to prove. A typical high-intersection scenario is illustrated below.

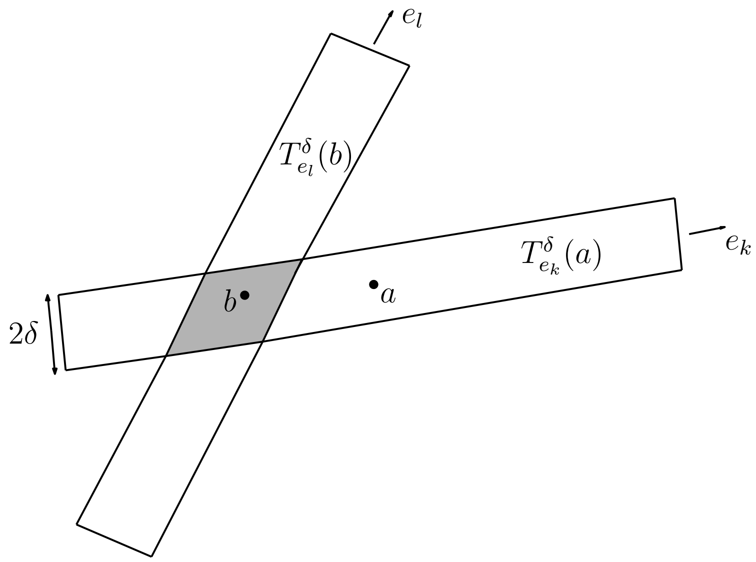

Figure 4.1: A typical high intersection scenario of two tubes in $\mathbb{R}^2$.

We begin by proving (4.1). Recall that we may assume all directions lie within $\frac{1}{10}$ of the vertical. We can easily reduce to the case where the two tubes have a common center at the origin. To see this, pick any $x \in T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b)$; we claim that

$$T^\delta_{e_k}(a) \subset 2 \cdot T^\delta_{e_k}(x) \text{ and } T^\delta_{e_l}(b) \subset 2 \cdot T^\delta_{e_l}(x). \tag{4.3}$$

Indeed, for any $y \in T^\delta_{e_k}(a)$ we have

$$|(y-x) \cdot e_k| \leq |(y-a) \cdot e_k| + |(x-a) \cdot e_k| \leq 1$$

since $x \in T^\delta_{e_k}(a)$ by assumption. Similarly, for any unit vector $v \perp e_k$, we have

$$|(y-x) \cdot v| \leq |(y-a) \cdot v| + |(x-a) \cdot v| \leq 2\delta$$

. A completely symmetric argument verifies the second containment in (4.3). Without loss of generality we may translate the system so that $x = 0$; thus we see that

$$T^\delta_{e_k}(a) \cap T^\delta_{e_l}(b) \subset 2 \cdot T^\delta_{e_k}(0) \cap 2 \cdot T^\delta_{e_l}(0),$$

so it suffices to show (4.1) with $a = b = 0$.

Let $x \in T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)$. We are going to parameterize x as we did in Chapter 3, only here we will have two parameterizations, one for each tube that x lies in. Accordingly, we know that x lies on some unit line segment I_k that is parallel to e_k with midpoint $m_k \in B(0, \delta)$ lying within the unique hyperplane through the origin with normal vector e_k; similarly, we know that x lies on some other unit line segment I_l that is parallel to e_l with midpoint $m_l \in B(0, \delta)$ lying within the unique hyperplane through the origin with normal vector e_l. This is to be absolutely precise, but an appeal to the figure below should make the situation more clear.

So we have two parameterizations of $x \in T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)$:

$$x = m_k + t_x e_k \text{ for some } |t_x| \leq \frac{1}{2} \tag{4.4}$$

$$x = m_l + t'_x e_l \text{ for some } |t'_x| \leq \frac{1}{2}. \tag{4.5}$$

Now t_x and t'_x are both fixed numbers so we may assume that $|t_x| \geq |t'_x|$. We want to bound $|x|$ from above, so appealing to (4.4) we see that it suffices to bound $|m_k|$ and $|t_x|$. We already have $|m_k| \leq \delta$, so we only need a bound for $|t_x|$.

Consider the triangle with vertices at m_k, m_l, and x as in the figure above. By the law of sines, we have that

$$\frac{\sin\theta}{|m_k - m_l|} = \frac{\sin\theta'}{|t_x|};$$

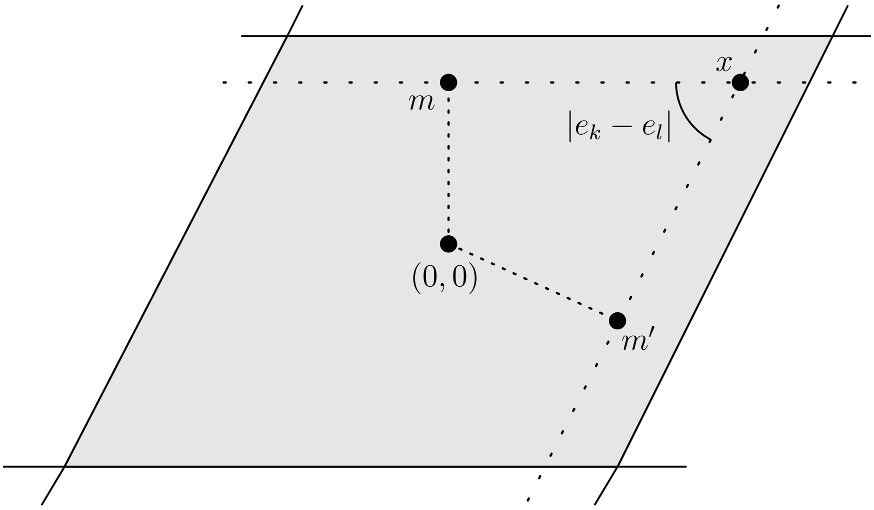

Figure 4.2: *A point $x \in T^\delta_{e_k} \cap T^\delta_{e_l}$ represented as the point of intersection of two line segments with orientations e_k and e_l, contained in their respective tubes.*

consequently, we see that

$$|t_x| \leq \frac{|m_k - m_l|}{\sin\theta} \lesssim \frac{\delta}{|e_k - e_l|}$$

since $\sin\theta \sim \theta = |e_k - e_l|$. Evidently then, we have

$$|x| \leq |m_k| + |t_x| \lesssim \delta + \frac{\delta}{|e_k - e_l|} \lesssim \frac{\delta}{|e_k - e_l|}. \tag{4.6}$$

The point x was arbitrarily chosen to lie in the intersection $T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)$, so (4.6) holds for all points in this intersection. Thus, we have

$$\operatorname{diam}(T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)) = \sup_{x \in T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)} 2|x| \lesssim \frac{\delta}{|e_k - e_l|}$$

by (4.6), verifying the first part of the lemma, (4.1).

For the second part of the lemma, we apply another containment trick. We claim that

$$T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0) \subset \frac{C\delta}{|e_k - e_l|} \cdot T^{|e_k - e_l|}_{e_k}(0)$$

for some constant C. But this is immediate from our work above. Indeed, since $x \in T^\delta_{e_k}(0)$, we know that $|x \cdot v| \leq \delta$ for any unit vector $v \perp e_k$ and by (4.1), we know that $x \in B(0, \frac{C\delta}{|e_k - e_l|})$ for some constant C. Thus,

$$|T^\delta_{e_k}(0) \cap T^\delta_{e_l}(0)| \leq \left| \frac{C\delta}{|e_k - e_l|} \cdot T^{|e_k - e_l|}_{e_k}(0) \right| \sim \frac{\delta}{|e_k - e_l|} \cdot \delta^{n-1} = \frac{\delta^n}{|e_k - e_l|}$$

as claimed. □

Córdoba's L^2 estimate exploits a duality argument. His original proof has a classical flavor as he explicitly analyzes the adjoint of the maximal operator; we will follow Wolff's treatment [33] since its methods are more geometrically transparent and also more prescient of later methods we will develop in the subsequent chapters.

Lemma 4.1.2. *Let $1 < p < \infty$ and let p' be the dual exponent of p. Suppose p has the following property: if $\{e_k\} \subset S^{n-1}$ is a maximal δ-separated set, and if $\delta^{n-1} \sum_k |y_k|^{p'} = 1$, then for any choice of points $a_k \in \mathbb{R}^n$ we have*

$$||\sum_k y_k \chi_{T^\delta_{e_k}(a_k)}||_{p'} \leq A. \tag{4.7}$$

Then there is a bound

$$||f^*_\delta||_{L^p(S^{n-1})} \lesssim A||f||_p. \tag{4.8}$$

Proof. Let $\{e_k\}$ be a maximal δ-separated subset of S^{n-1} and suppose that p has the property stated in the hypothesis of the lemma. We can discretize the domain of the Kakeya maximal operator by appealing to a doubling-type argument. To see this, suppose $e, e' \in S^{n-1}$ are such that $|e - e'| < \delta$ and fix any $x \in T^\delta_e(a)$ where a is an arbitrary point in $\mathbb{R}^n$. Let E denote the principal axis of $T^\delta_e(a)$ and let $b \in E$ be such that $||x - b|| = \inf_{y \in E} ||x - y||$. This defines b to be the unique point in E closest to x (we use the norm notation here to emphasize that the distance is measured in the standard Euclidean way and to avoid confusion with our notation for the metric on S^{n-1}). We will exploit the simple fact that the Euclidean distance between two points is bounded by the sum of the distances between these two points and any intermediate point.

Consider the tube $T^\delta_{e'}(a)$; we similarly let E' denote its axis and define $b' \in E'$ to satisfy $||b - b'|| = \inf_{y \in E'} ||b - y||$. Now since the two tubes have the same center, we know that

$$||b - b'|| = ||a - b|| \sin(|e - e'|) \lesssim |e - e'| < \delta$$

(see the figure on the next page). Thus, by the triangle inequality,

$$||x - b'|| \leq ||x - b|| + ||b - b'|| \lesssim \delta,$$

and so $x \in T^{C\delta}_{e'}(a)$ for some constant C.

We have shown that any tube in $\mathbb{R}^n$ can be covered by a bounded number of equidimensional tubes with a different orientation so long as this new orientation is not very far from the original. Of course, this agrees with our intuition and we can immediately see that this implies $f^*_\delta(e) \lesssim f^*_{C\delta}(e')$ provided $|e - e'| < \delta$.

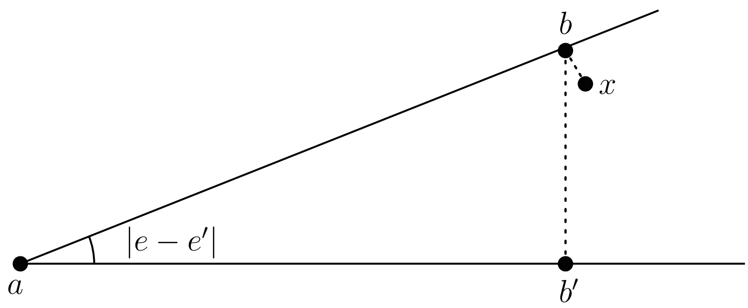

Figure 4.3: *The distance between a point $x \in T_e^\delta(a)$ to another tube $T_{e'}^\delta(a)$.*

We use the above procedure now to discretize the domain of our maximal operator and write

$$\begin{aligned} ||f_\delta^*||_p &\leq \left(\sum_k \int_{D(e_k,\delta)} f_{C\delta}^*(e)^p de \right)^{1/p} \\ &\lesssim \left(\delta^{n-1} \sum_k f_{C\delta}^*(e_k)^p \right)^{1/p} . \end{aligned}$$

Using duality between l_p and $l_{p'}$ we have

$$||f_\delta^*||_p \lesssim \delta^{n-1} \sum_k y_k f_{C\delta}^*(e_k)$$

for some sequence $\{y_k\}$ such that $\delta^{n-1} \sum_k |y_k|^{p'} = 1$. Plugging in the definition of the maximal function, we find that

$$\begin{aligned} ||f_\delta^*||_p &\lesssim \delta^{n-1} \sum_k y_k \frac{1}{|T_{e_k}^{C\delta}(a_k)|} \int_{T_{e_k}^{C\delta}(a_k)} f(x) dx \\ &\lesssim \delta^{n-1} \sum_k y_k \frac{1}{|T_{e_k}^{\delta}(a_k)|} \int_{T_{e_k}^{\delta}(a_k)} f(x/C) dx \end{aligned}$$

for some choice of $\{a_k\}$. Notice that the last inequality follows via the general containment fact that $T^{C\delta} \subset C \cdot T^\delta$. Now, since $|T_{e_k}^\delta(a_k)| \sim \delta^{n-1}$, it follows that

$$\begin{aligned} ||f_\delta^*||_p^p &\lesssim \left[\int \left(\sum_k y_k \chi_{T_{e_k}^\delta(a_k)}(x) \right) f(x/C) dx \right]^p \\ &\lesssim || \sum_k y_k \chi_{T_{e_k}^\delta(a_k)} ||_{p'}^p \cdot ||f||_p^p \end{aligned}$$

by Hölder's inequality. Applying (4.7) and taking pth roots[2], we arrive at the bound (4.8) as claimed. ☐

We proceed to the main result on $\mathbb{R}^2$.

Proof of Theorem 4.0.1. Using the previous lemma, it suffices to show that for any sequence $\{y_k\}$ with $\delta \sum_k y_k^2 = 1$ and any maximal δ-separated subset $\{e_k\}$ of S^1 we have

$$|| \sum_k y_k \chi_{T^\delta_{e_k}(a_k)} ||_2 \lesssim \sqrt{\log \frac{1}{\delta}}. \tag{4.9}$$

We apply the above geometrical lemma to reduce the left-hand side to

$$\begin{aligned} || \sum_k y_k \chi_{T^\delta_{e_k}(a_k)} ||_2^2 &= \sum_{k,l} y_k y_l |T^\delta_{e_k}(a_k) \cap T^\delta_{e_l}(a_l)| \\ &\lesssim \sum_{k,l} y_k y_l \frac{\delta^2}{|e_k - e_l|}, \end{aligned}$$

which by a regrouping becomes

$$\lesssim \sum_{k,l} \sqrt{\delta} y_k \sqrt{\delta} y_l \frac{\delta}{|e_k - e_l|}. \tag{4.10}$$

We apply Cauchy-Schwarz to estimtate this quantity further; thus (4.10) becomes

$$\begin{aligned} \sum_{k,l} \sqrt{\delta} y_k \left(\frac{\delta}{|e_k - e_l|} \right)^{1/2} \sqrt{\delta} y_l \left(\frac{\delta}{|e_k - e_l|} \right)^{1/2} &\le \left(\sum_{k,l} \delta y_k^2 \frac{\delta}{|e_k - e_l|} \right)^{1/2} \cdot \left(\sum_{k,l} \delta y_l^2 \frac{\delta}{|e_k - e_l|} \right)^{1/2} \\ &= \sum_{k,l} \delta y_k^2 \frac{\delta}{|e_k - e_l|} \\ &\lesssim \log \frac{1}{\delta} \sum_k \delta y_k^2 \\ &= \log \frac{1}{\delta}. \end{aligned}$$

The second to last line follows from the fact that $|e_k - e_l| \lesssim 1$ and that all directions in our collection our maximally δ-separated; indeed, we see then for any fixed k that

$$\sum_l \frac{\delta}{|e_k - e_l|} \lesssim \sum_{1 \le l \le \frac{1}{\delta}} \frac{\delta}{l\delta} = \sum_{1 \le l \le \frac{1}{\delta}} \frac{1}{l} \sim \log \frac{1}{\delta}.$$

Thus we have (4.9), proving the theorem. ☐

[2]This pth root is the only time our image space comes into play. Notice that the same reasoning will hold if we were after an $L^p \to L^q$ estimate as well.

Notice that Córdoba's estimate does indeed settle the Kakeya maximal function conjecture in $\mathbb{R}^2$; for every $\epsilon > 0$ there exists a constant C_ϵ such that $\log \frac{1}{\delta} \leq C_\epsilon \delta^{-\epsilon}$ and thus

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||f_\delta^*||_2 \leq C_\epsilon \delta^{-\epsilon} ||f||_2.$$

By Lemma 2.3, we also know that Córdoba's estimate tells us that Kakeya sets in $\mathbb{R}^2$ must have full Hausdorff dimension (and so full Minkowski dimension as well).

4.2 L^2 Estimates in $\mathbb{R}^n$

Although Córdoba's result was originally only obtained in $\mathbb{R}^2$, it can be easily extended to an L^2 estimate on $\mathbb{R}^n$ for $n \geq 3$. Instead of just directly generalizing the above proof, we offer an alternative argument due to Bourgain [6]. This proof is purely Fourier analytic and has an elegant appeal. As with Córdoba's proof, we will need a geometrical lemma to obtain the desired result. The argument will force us to again consider a set of tubes, but this time we will utilize a convolution operation so that they are all centered at the origin. A scaling will also make these tubes quite fat instead of thin.

Lemma 4.2.1 (Bourgain)**.** *Let $C \geq 1$ be some constant and suppose $\delta \ll C$. For any fixed $\xi \in \mathbb{R}^n$, one has the bound*

$$|\{e \in S^{n-1} : C \cdot T_e^{1/\delta}(0) \ni \xi\}| \lesssim \frac{1}{1 + |\xi|}.$$

Notice that here, the tube $C \cdot T_e^{1/\delta}(0)$ is oriented with the shortest axis in the e direction.

Basically, this lemma gives a quantitative estimate on the size of the set of directions that define $C \cdot T_e^{1/\delta}(0)$ tubes or slabs containing the fixed point $\xi \in \mathbb{R}^n$. These fat slabs appear naturally as the dual objects to thin tubes via the Fourier transform and indeed this is how they will appear in our main result. However, we first provide a proof of this geometrical lemma.

Proof. First, suppose $|\xi| \leq 10C$ so that $\frac{1}{1+10C} \leq \frac{1}{1+|\xi|}$. It follows trivially then that

$$|\{e \in S^{n-1} : C \cdot T_e^{1/\delta}(0) \ni \xi\}| \leq |S^{n-1}| \lesssim \frac{1}{1 + 10C} \leq \frac{1}{1 + |\xi|}.$$

If instead $|\xi| > \frac{C}{\delta}$, then $\xi \notin C \cdot T_e^{1/\delta}(0)$ for all $e \in S^{n-1}$ and so the estimate is again trivial.

Lastly, if $10C < |\xi| \leq \frac{C}{\delta}$, then to ensure that $\xi \in C \cdot T_e^{1/\delta}(0)$ for some e, the vectors ξ and e must be nearly orthogonal. By definition, we have

$$C \cdot T_e^{1/\delta}(0) = \{\eta \in \mathbb{R}^n : |\eta \cdot e| \leq C,\ |\eta \cdot v| \leq \frac{C}{\delta} \text{ for all } v \perp e\}. \qquad (4.11)$$

We may rotate our system so that ξ lies on the first standard Euclidean axis, i.e. so that $\xi = (\xi_1, 0, \ldots, 0)$. Now using our above description of the slab, notice that

$$\{e \in S^{n-1} : C \cdot T_e^{1/\delta}(0) \ni \xi\} = \{e : |\xi \cdot e| \le C,\ |\xi \cdot v| \le \frac{C}{\delta}\ \forall\, v \perp e\}$$
$$= \left\{ e : \left| \frac{\xi}{|\xi|} \cdot e \right| \le \frac{C}{|\xi|},\ \left| \frac{\xi}{|\xi|} \cdot v \right| \le \frac{C}{\delta |\xi|}\ \forall\, v \perp e \right\}. \quad (4.12)$$

The second inequality in (4.12) is trivial; indeed, for any $e \in S^{n-1}$ and any $v \perp e$,

$$\left| \frac{\xi}{|\xi|} \cdot v \right| \le \left| \frac{\xi}{|\xi|} \right| \cdot |v| = 1 \le \frac{C}{\delta |\xi|},$$

since $|\xi| \le \frac{C}{\delta}$. So (4.12) reduces to simply

$$\left\{ e \in S^{n-1} : \left| \frac{\xi}{|\xi|} \cdot e \right| \le \frac{C}{|\xi|} \right\}. \quad (4.13)$$

Now $\left| \frac{\xi}{|\xi|} \cdot e \right| = |e_1|$ where $e = (e_1, \ldots, e_n)$, so we see that the remaining restriction in (4.13) applies only in a single degree of freedom. More precisely, for any direction e in the set defined by (4.13), there is no restriction on its $e_2, \ldots, e_n$ components; the only restriction we need make is that $|e_1| \le \frac{C}{|\xi|}$. This defines a subset of the sphere that looks much like a ring with thickness $\sim \frac{1}{|\xi|}$ in the e_1 direction. Consequently, the size of this ring is bounded by $\lesssim \frac{1}{|\xi|} \lesssim \frac{1}{1+|\xi|}$, since $|\xi| \gtrsim 1$. The lemma now follows accordingly. □

We are now ready to prove our main theorem.

Theorem 4.2.2 (Bourgain). *Let $n \ge 3$. Then we have the L^2 estimate*

$$||f_\delta^*||_{L^2(S^{n-1})} \lesssim \delta^{-\frac{n-2}{2}} ||f||_{L^2(\mathbb{R}^n)}.$$

Proof. We begin by writing our maximal operator as a maximum of convolutions against smooth functions. Naturally, this will allow us to take advantage of some nice properties in the Fourier space.

Let $\varphi_e^\delta(x) = \frac{1}{\delta^{n-1}} \chi_{T_e^\delta(0)}(x)$. Then

$$f_\delta^*(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{\delta^{n-1}} \int_{T_e^\delta(a)} f(y)dy$$
$$= \sup_{a \in \mathbb{R}^n} \frac{1}{\delta^{n-1}} \int_{\mathbb{R}^n} \chi_{T_e^\delta(0)} f(y-a)dy$$
$$= \sup_{a \in \mathbb{R}^n} (\varphi_e^\delta * f)(a).$$

The above convolution is against a rough cut-off function, but we can majorize this by a smooth representation to take better advantage of certain properties of the Fourier transform. Choose any function $\phi \in \mathcal{S}(\mathbb{R})$, $\phi \geq 0$ such that $\hat{\phi}$ has compact support, say in the interval $[-\frac{C}{2}, \frac{C}{2}]$, and $\phi(x) \geq 1$ when $|x| \leq 1$. Define $\psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(x) = \phi(x_1) \frac{1}{\delta^{n-1}} \prod_{j=2}^{n} \phi\left(\frac{x_j}{\delta}\right)$$

where we have written $x \in \mathbb{R}^n$ as $x = (x_1, x_2, \ldots, x_n)$.

Let e_1 denote the first canonical basis vector in $\mathbb{R}^n$. We claim that $\varphi_{e_1}^{\delta} \leq \psi$. To see this, notice that if $x \notin T_{e_1}^{\delta}(0)$, then $\varphi_{e_1}^{\delta}(x) = 0 \leq \psi(x)$. If instead $x \in T_{e_1}^{\delta}(0)$, then $\phi(x_1) \geq 1$ and $\phi\left(\frac{x_j}{\delta}\right) \geq 1$ for all $2 \leq j \leq n$ and so the inequality follows by definition. Thus we have

$$f_\delta^*(e_1) \leq \sup_{a \in \mathbb{R}^n} (\psi * f)(a).$$

Let ρ_e denote the rotation that sends the unit vector e to e_1. Define $\psi_e = \psi \circ \rho_e$ and notice that $\varphi_e^{\delta} \leq \psi_e$ by the same reasoning as before. Consequently,

$$f_\delta^*(e) \leq \sup_{a \in \mathbb{R}^n} (\psi_e * f)(a) = ||\psi_e * f||_\infty. \tag{4.14}$$

We proceed to the analysis. First we estimate (4.14) as

$$\begin{aligned} ||\psi_e * f||_\infty &\leq ||\hat{\psi}_e \hat{f}||_1 \\ &= \int_{\mathbb{R}^n} |\hat{\psi}_e(\xi)| \cdot |\hat{f}(\xi)| d\xi \end{aligned} \tag{4.15}$$

which follows by the trivial fact that the norm of the Fourier transform defined as an operator from L^1 to L^∞ is bounded by 1. Rewriting the righthand side of (4.15) and applying Cauchy-Schwarz, we find that

$$\begin{aligned} f_\delta^*(e) &\leq \int \left(|\hat{f}(\xi)| \sqrt{|\hat{\psi}_e(\xi)| \cdot (1 + |\xi|)} \right) \left(\sqrt{\frac{|\hat{\psi}_e(\xi)|}{1 + |\xi|}} \right) d\xi \\ &\leq \left(\int |\hat{\psi}_e(\xi)| \cdot |\hat{f}(\xi)|^2 (1 + |\xi|) d\xi \right)^{1/2} \left(\int \frac{|\hat{\psi}_e(\xi)|}{1 + |\xi|} d\xi \right)^{1/2}. \end{aligned} \tag{4.16}$$

We turn our attention to estimating the second integral in (4.16). To do this, we first need to estimate $|\hat{\psi}_e|$. In fact, since $\hat{\psi}_e = \hat{\psi} \circ \rho_e$ we need only analyze $\hat{\psi}$. Clearly, $|\hat{\psi}|$ is bounded universally by some constant. By a direct computation, we also have that

$$\hat{\psi}(\xi) = \hat{\phi}(\xi_1) \prod_{j=2}^{n} \hat{\phi}(\delta \xi_j).$$

By our assumptions on $\hat{\phi}$ we see that $\hat{\psi}$ is supported on a tube of size $C(1\times 1/\delta\times\ldots\times 1/\delta)$ centered at the origin and oriented so that the shortest dimension points in the e_1-direction. It now follows directly that $|\hat{\psi}_e|$ is bounded and that $\hat{\psi}_e$ is supported on a tube of size $C(1\times 1/\delta\times\ldots\times 1/\delta)$ centered at the origin and oriented so that the shortest dimension points in the e-direction, i.e. $\hat{\psi}_e$ is supported on $C\cdot T_e^{1/\delta}(0)$. Then

$$\int_{\mathbb{R}^n}\frac{|\hat{\psi}_e(\xi)|}{1+|\xi|}d\xi\lesssim\int_{C\cdot T_e^{1/\delta}(0)}\frac{d\xi}{1+|\xi|}=\int_{C\cdot T_{e_1}^{1/\delta}(0)}\frac{d\xi}{1+|\xi|},$$

where the last equality follows by making the obvious unitary transformation in the integral to rotate our region of integration. We use polar coordinates, with $\xi=(\xi_1,\xi')=(\xi_1,\xi_2,\ldots,\xi_n)$, to rewrite the final integral above and estimate

$$\begin{aligned}\int_{|\xi_1|\le C,\ |\xi_j|\le\frac{C}{\delta}}\frac{d\xi d\xi'}{1+|\xi|}&\lesssim\int_{|\xi_j|\le\frac{C}{\delta}}\frac{d\xi'}{1+|\xi'|}\\&\lesssim\int_0^{\frac{C}{\delta}}\frac{r^{n-2}dr}{r}\\&\lesssim\int_0^{\frac{C}{\delta}}r^{n-3}dr\\&\sim\delta^{-(n-2)}.\end{aligned}\tag{4.17}$$

Notice that this is the only part of the argument that would need to be altered if we let $n=2$. Obviously, the bound would then become $\log(1/\delta)$, agreeing with the bound we acquired in the previous section.

The estimation done, we can now prove the desired L^2 bound. Combining (4.16) and (4.17), we find that

$$||f_\delta^*||_2^2\lesssim\delta^{-(n-2)}\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2(1+|\xi|)\left(\int_{S^{n-1}}|\hat{\psi}_e(\xi)|de\right)d\xi.\tag{4.18}$$

Now we are in a position to apply the lemma. Notice that for any fixed $\xi\in\mathbb{R}^n$,

$$\int_{S^{n-1}}|\hat{\psi}_e(\xi)|de\lesssim|\mathrm{supp}(\hat{\psi}_e(\xi))|=|\{e\in S^{n-1}:C\cdot T_e^{1/\delta}(0)\ni\xi\}|\lesssim\frac{1}{1+|\xi|}$$

by Lemma 4.2.1. Thus (4.18) becomes

$$||f_\delta^*||_2^2\lesssim\delta^{-(n-2)}\int_{\mathbb{R}^n}|\hat{f}(\xi)|^2d\xi=\delta^{-(n-2)}||f||_2^2$$

by Plancherel's theorem. Taking square roots, we obtain Theorem 4.2.2. □

This result effectively illuminates the L^2 behavior of our maximal operator, but clearly we would like more. In the next two sections we will build up some impressive machinery to tackle the problem of L^p estimation of the Kakeya operator on a general $\mathbb{R}^n$. Most of the ideas that follow are due to Bourgain [6], Tao [29] and Wolff [31] and have many wide-ranging applications than just directly to the problem of Kakeya. The techniques will utitlize every component we have so far presented and in their totality represent a truly sophisticated piece of analytical craftsmanship.

Chapter 5

The Bush Argument

The main result of Córdoba in Chapter 4 gave us the estimate $\mathcal{K}(2)$, in the notation of Section 2.4. In this chapter, we present Bourgain's so-called "bush" argument which provides the improved estimate $\mathcal{K}(\frac{n+1}{2})$ and so the lower bound of $\frac{n+1}{2}$ on the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$. We will give two proofs of this estimate. The analyses will be very similar, but the preliminary reductions will differ enough to be instructive. In Section 5.1, we give Bourgain's [6] original proof, following Wolff's treatment [33], and prove the restricted weak type-$(\frac{n+1}{2}, n+1)$ estimate

Theorem 5.0.1 (Bourgain)**.**

$$||f^*_\delta||_{n+1,\infty} \lesssim_n \delta^{-\frac{n-1}{n+1}} ||f||_{\frac{n+1}{2},1}. \tag{5.1}$$

As we have seen, this is more than we need to get the analogous bound on the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$, as it implies the equivalent strong type estimate on the diagonal (see Section 2.4). We feel that this analytical overkill is still instructive though due to the prevalency of these techniques in much of the literature from the 1990's.

In Section 5.2, we will once again prove the estimate $\mathcal{K}(\frac{n+1}{2})$, this time exploiting a similar duality argument as was used in Section 4.1. However, we will refine this approach and thus, streamline the analysis. This will not only illustrate the different philosophies behind deriving Kakeya estimates, but it will lay the foundation for the much more complicated arguments of Chapter 6.

The main strategy of the bush argument will involve bounding the size of a set E in $\mathbb{R}^n$ from below by considering its intersection with a δ-separated family of tubes. The δ-separation will allow us to control the intersection and thus estimate $|E|$ from below by essentially the complement of this intersection. If we think of many lines intersecting at a single point, then expanding these lines to small tubes should not produce a large intersection. Thus, when many tubes intersect E at a certain point, this technique should give a good lower bound by virtue of the intersection being quite small.

5.1 Bourgain's Approach

Equation (5.1) is a restricted weak type-$(\frac{n+1}{2}, n+1)$ estimate and as such should be interpreted as

$$|\{e \in S^{n-1} : (\chi_E)^*_\delta(e) \geq \lambda\}| \leq \left(C_n \delta^{-\frac{n-1}{n+1}} \frac{|E|^{\frac{2}{n+1}}}{\lambda} \right)^{n+1}, \tag{5.2}$$

for all sets $E \subset \mathbb{R}^n$ with finite measure, all $0 < \lambda \leq 1$ and where $p = \frac{n+1}{2}$, $q = n+1$. We make the following reduction.

Lemma 5.1.1. *Let E be a set in $\mathbb{R}^n$ and for $0 < \lambda \leq 1$, denote by $D_E(\lambda) = \{e \in S^{n-1} : (\chi_E)^*_\delta(e) \geq \lambda\}$. Suppose that the subset $\{e_j\}_{j=1}^M \subset D_E(\lambda)$ is δ-separated. If*

$$|E| \gtrsim \delta^{n-1} \lambda^{\frac{n+1}{2}} \sqrt{M}, \tag{5.3}$$

then equation (5.2) *holds.*

The lemma will follow from a simple combinatorial fact about δ-separated directions. For this, we make a definition.

Definition 5.1.2. *For any subset $\Omega \subseteq S^{n-1}$, $|\Omega| > 0$, $r > 0$, define its **r-entropy** $\mathcal{N}_r(\Omega)$ as the maximum possible cardinality for an r-separated subset of Ω.*

Fact 5.1.3. *In the notation just defined,*

$$\frac{1}{r^{n-1}} \gtrsim \mathcal{N}_r(\Omega) \gtrsim \frac{|\Omega|}{r^{n-1}}. \tag{5.4}$$

Proof. To see this, we consider Ω and S^{n-1} to be hypersurfaces in $\mathbb{R}^n$. If we cover the sphere by connected pieces or caps, each of radius about r^{n-1}, so that the center of any particular cap is not contained in any others, then we see $\mathcal{N}_r(S^{n-1}) \sim \frac{|S^{n-1}|}{r^{n-1}} \sim \frac{1}{r^{n-1}}$. Now, for any other subset $\Omega \subset S^{n-1}$, we clearly have $\mathcal{N}_r(\Omega) \leq \mathcal{N}_r(S^{n-1})$ and so the first inequality in (5.4) follows.

Now suppose that the second inequality does not hold for some $\Omega \subset S^{n-1}$ such that $|\Omega| > 0$. Then since we can always cover S^{n-1} by a finite number of rotated copies of Ω, we see that we would have to contradict our first inequality for $\Omega = S^{n-1}$. □

We can now exploit this fact to prove the lemma.

Proof of Lemma 5.1.1. It suffices to prove the lemma for $M = \mathcal{N}_\delta(D_E(\lambda))$; by equation (5.4) and our hypothesis (5.3), we have

$$\delta^{n-1}|D_E(\lambda)| \lesssim \delta^{2(n-1)} M \lesssim \frac{|E|^2}{\lambda^{n+1}}.$$

Rearranging this expression, we arrive at (5.2). □

So we have reduced proving the pure restricted weak type estimate of (5.2) to only having to prove (5.3) for any δ-separated subset $\{e_j\}_{j=1}^{M} \subset D_E(\lambda)$.

To apply Lemma 5.1.1, we want a lower bound on the size of any set $E \subset \mathbb{R}^n$, $|E| > 0$, given some δ-separated subset $\{e_j\}_{j=1}^{M} \subseteq D_E(\lambda)$. Using the definition of the Kakeya maximal function, we see that for any $1 \leq j \leq M$,

$$(\chi_E)_\delta^*(e_j) \geq \frac{|E \cap T_{e_j}^\delta|}{|T_{e_j}^\delta|} \geq \lambda, \tag{5.5}$$

for some δ-tube oriented in the e_j direction. In this way, we may assign a tube $T_{e_j}^\delta$ with a relatively large E-intersection to each direction e_j in our δ-separated subset of $D_E(\lambda)$. Analyzing how these tubes interact with the set E will lead us to the lower bound necessary to apply the lemma.

Fix a positive integer μ to be determined later; there are two basic cases to consider. First, our tubes may be well dispersed so that no point of E belongs to more than μ tubes. For this case, we use a trivial lower bound. The other alternative is that there may be some point in E that belongs to many different, say more than μ, tubes. This case will lead us to consider how these tubes intersect at the high multiplicity point. If many tubes intersect a single point of E, then since their directions are all δ-separated, the object formed by these tubes will look like a spiky burr or bush; thus, we will be forced to analyze the intersection properties of this bush object. This second case will require the other facts that we have established in addition to two more general lemmas in order to attain a suitable lower bound on $|E|$.

Inequality (5.5) tells us that if $e_j \in D_E(\lambda)$, then we have the bound $|E \cap T_{e_j}^\delta| \gtrsim \lambda\delta^{n-1}$. This next lemma tells us that the same estimate will hold if we cut out a small enough portion of the intersection.

Lemma 5.1.4. *Let $E \subset \mathbb{R}^n$ be such that $|E| > 0$. Let $0 < \lambda \leq 1$ and suppose we have a tube T_e^δ such that $|E \cap T_e^\delta| \geq \lambda |T_e^\delta|$. Then for any $a \in E \cap T_e^\delta$, there exists a constant c_0 dependent on λ, δ and e such that*

$$|E \cap T_e^\delta \cap B(a, c_0^{-1}\lambda)^c| \geq \frac{\lambda}{2}|T_e^\delta|, \tag{5.6}$$

where A^c denotes the complement of the set A in $\mathbb{R}^n$.

Proof. Pick a point $a \in E \cap T_e^\delta$. For any choice of c_0, we can write

$$|E \cap T_e^\delta| = |E \cap T_e^\delta \cap B(a, c_0^{-1}\lambda)| + |E \cap T_e^\delta \cap B(a, c_0^{-1}\lambda)^c|. \tag{5.7}$$

We want to bound the first term in (5.7) from above. Observe that

$$|E \cap T_e^\delta \cap B(a, c_0^{-1}\lambda)| \leq |T_e^\delta \cap B(a, c_0^{-1}\lambda)|;$$

so choose c_0 (dependent on λ, δ and e) such that $|T_e^\delta \cap B(a, c_0^{-1}\lambda)| \leq \frac{\lambda}{2}|T_e^\delta|$. Explicitly, this can be accomplished as follows. If $\lambda \leq c_0\delta$, then we trivially have

$$|T_e^\delta \cap B(a, c_0^{-1}\lambda)| \leq |B(a, c_0^{-1}\lambda)| \sim \left(\frac{\lambda}{c_0}\right)^n \leq \frac{\lambda}{c_0}\delta^{n-1} \sim \frac{\lambda}{2}|T_e^\delta|.$$

If instead $c_0\delta < \lambda \leq 1$, then we contain the intersection $T_e^\delta \cap B(a, c_0^{-1}\lambda)$ in a tube centered at a of length $\sim c_0^{-1}\lambda$ in the e direction and width δ in all orthogonal directions. Consequently, the size of the intersection is dominated by the size of the tube which can itself be dominated by $\frac{\lambda}{2}|T_e^\delta|$ for an appropriately large c_0.

Combining the above analysis with our hypothesis that $|E \cap T_e^\delta| \geq \lambda|T_e^\delta|$ and (5.7), we obtain

$$|E \cap T_e^\delta \cap B(a, c_0^{-1}\lambda)^c| \geq \lambda|T_e^\delta| - |T_e^\delta \cap B(a, c_0^{-1}\lambda)| \geq \frac{\lambda}{2}|T_e^\delta|,$$

as was to be shown. □

This bound on the trimmed intersection will allow us to construct a lower bound for $|E|$ by considering many such intersections that are disjoint by virtue of the trimming action. This is highly suggestive of the case when many tubes intersect E at or near a single point and indeed, this is how we will utilize Lemma 5.1.4 in the bush argument. We have a final counting lemma before we begin the proof of Bourgain's theorem.

Lemma 5.1.5. *Suppose $\{e_j\}_{j=1}^m \subset S^{n-1}$ is δ-separated and let c_1 be some constant to be determined later. Then for any fixed k, we have*

$$\#\left\{j : \theta(e_j, e_k) \geq c_1\frac{\delta}{\lambda}\right\} \gtrsim \lambda^{n-1}m. \tag{5.8}$$

Proof. Clearly we need to make a combinatorial argument to quantify the cardinality of the set in (5.8); this is where our Fact 5.1.3 will come into play. Let $C(e_j, \delta)$ denote the cap on S^{n-1} with center e_j and radius δ; then

$$\{e_j\}_{j=1}^m \subset \bigcup_{j=1}^m C(e_j, \delta),$$

a disjoint union of δ-caps. For convenience, denote this union by U. To prove (5.8), we need a lower bound on a maximal $c_1\frac{\delta}{\lambda}$-separation of $\{e_j\}_{j=1}^m$. We claim that the size of such a separation is at least as big as the maximum cardinality of a $2c_1\frac{\delta}{\lambda}$-separation of U, which, by (5.4), is bounded by

$$\mathcal{N}_{2c_1\frac{\delta}{\lambda}}(U) \gtrsim \frac{m\delta^{n-1}}{(2c_1\delta/\lambda)^{n-1}} \sim \lambda^{n-1}m. \tag{5.9}$$

By construction, whatever elements of U we pick for our maximal $2c_1\frac{\delta}{\lambda}$-separated subset, we cannot have any two of these lying in the same cap. So in a sense, we are in fact $2c_1\frac{\delta}{\lambda}$-separating our caps. Notice however that we cannot immediately assert that the centers of these caps are separated in exactly the same way, as the figure below demonstrates.

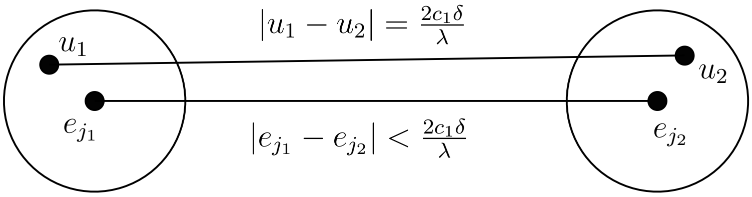

Figure 5.1: *Two $\frac{2c_1\delta}{\lambda}$-separated directions $u_1, u_2 \in U$ lying in caps with centers e_{j_1}, e_{j_2} that are not $\frac{2c_1\delta}{\lambda}$-separated.*

What we can say with certainty is that the centers of these caps are not separated by much less. More precisely, we have

$$\theta(e_{j_1}, e_{j_2}) \geq 2c_1\frac{\delta}{\lambda} - c_2\delta, \tag{5.10}$$

where c_2 is the absolute constant such that $\operatorname{diam}(C(e,\delta)) = c_2\delta$. We can now determine that whatever the value of c_1 may be, we must insist that $c_1 > c_2$. With this choice, (5.10) becomes

$$\theta(e_{j_1}, e_{j_2}) \geq c_1\frac{\delta}{\lambda}.$$

Thus, any $2c_1\frac{\delta}{\lambda}$-separation of U automatically gives a $c_1\frac{\delta}{\lambda}$-separation of the discrete set $\{e_j\}_{j=1}^m$. More precisely, let $\{\alpha_r\}_{r=1}^R$ be a $2c_1\frac{\delta}{\lambda}$-separated subset of U. Pick α_r and $\alpha_{r'}$, $r \neq r'$. Then $\alpha_r \in C(e_{j_r}, \delta)$ and $\alpha_{r'} \in C(e_{j_{r'}}, \delta)$. So by the above analysis, we know that $\theta(e_{j_r}, e_{j_{r'}}) \geq c_1\frac{\delta}{\lambda}$; thus, $\{e_{j_r}\}_{r=1}^R$ is a $c_1\frac{\delta}{\lambda}$-separated subset of $\{e_j\}_{j=1}^m$. Since this procedure works for any $2c_1\frac{\delta}{\lambda}$-separation of U, including a maximal separation, we conclude that $\#\left\{j : \theta(e_j, e_k) \geq c_1\frac{\delta}{\lambda}\right\} \geq \mathcal{N}_{2c_1\frac{\delta}{\lambda}}(U)$ for any fixed k. Applying (5.9), it now follows that

$$\#\left\{j : \theta(e_j, e_k) \geq c_1\frac{\delta}{\lambda}\right\} \gtrsim \lambda^{n-1}\mu$$

for any fixed k. $\square$

With all the necessary lemmas established, we proceed to the proof of Bourgain's theorem.

Proof of Theorem 5.0.1. We analyze the two multiplicity cases separately, choosing a suitable μ upon completion.

Case 1. *(low multiplicity)* Suppose that no point of E belongs to more than μ of our tubes $T^\delta_{e_j}$, i.e.

$$\sum_{j=1}^{M} \chi_{T_{e_j}}(x) \leq \mu,$$

for all $x \in E$. Integrating over E, we have

$$|E| \geq \frac{1}{\mu} \sum_j |E \cap T^\delta_{e_j}|.$$

Applying equation (5.5) and summing over all j yields

$$|E| \gtrsim \frac{\lambda}{\mu} M \delta^{n-1}. \tag{5.11}$$

Notice that this estimate holds for any μ we choose; however, the estimate becomes worse the larger we make μ.

Case 2. *(high multiplicity)* Now suppose that some point $a \in E$ belongs to more than μ tubes $T^\delta_{e_j}$. Reordering if necessary, we may assume that $a \in T^\delta_{e_j}$ for all $j \leq \mu + 1$. As before, we want to utilize (5.5), but it does not suffice to just consider the intersections $E \cap T^\delta_{e_j}$ superficially. Instead, we will cut out a small portion of the intersection near the high multiplicity point and construct a lower bound for the remaining set. Applying Lemma 5.1.4, we have that there exists some constant c_0 such that

$$|E \cap T^\delta_{e_j} \cap B(a, c_0^{-1}\lambda)| \gtrsim \lambda \delta^{n-1} \tag{5.12}$$

for all $j \leq \mu + 1$.

Now we apply Lemma 4.1.1 (Córdoba's geometrical lemma) to convert (5.12) into a tighter lower bound for $|E|$. Fix $k \leq \mu + 1$; then if $j \leq \mu + 1$, the intersection $T^\delta_{e_j} \cap T^\delta_{e_k}$ contains a and has diameter $\lesssim \frac{\delta}{\theta(e_j, e_k)}$ by Lemma 4.1.1. Let c_1 be a suitable constant determined by applying Lemma 5.1.5 to the collection $\{e_j\}_{j=1}^{\mu+1}$ such that we also have $c_1 > c_0$. Notice if $\theta(e_j, e_k) \geq c_1 \frac{\delta}{\lambda}$, then we have $\mathrm{diam}(T^\delta_{e_j} \cap T^\delta_{e_k}) \lesssim c_0^{-1}\lambda$. Consequently, the sets $E \cap T^\delta_{e_j} \cap B(a, c_0^{-1}\lambda)^c$ and $E \cap T^\delta_{e_k} \cap B(a, c_0^{-1}\lambda)^c$ are disjoint. This is the crucial geometric fact for the bush argument and it is really just an extension of the obvious fact that two lines intersect at a single point. Of course though, we have had to adapt this bit of trivia to suit our δ-thickened needs. What this affords us is that we can now bound $|E|$ from below by the *sum* of certain trimmed intersections as in (5.12) rather than by just a single trimmed intersection, thus tightening the bound. What we have then is that

$$
\begin{aligned}
|E| &\geq \sum_{j:\theta(e_j,e_k)\geq c_1\frac{\delta}{\lambda}} |E \cap T^{\delta}_{e_j} \cap B(a, c_0^{-1}\lambda)^c| \\
&\gtrsim \# \left\{ j : \theta(e_j, e_k) \geq c_1 \frac{\delta}{\lambda} \right\} \cdot \lambda \delta^{n-1}. \qquad (5.13)
\end{aligned}
$$

Notice that we need not include the trimmed intersection corresponding to the tube $T^{\delta}_{e_k}$ since this would not tighten the bound by more than an adjustment to the implicit constant. The cardinality of the set in (5.13) is of the form treated in Lemma 5.1.5, so combining (5.13) with the estimate (5.8), we arrive at the bound

$$
|E| \gtrsim \lambda^n \delta^{n-1} \mu, \qquad (5.14)
$$

completing the analysis of the high multiplicity case.

It is clear that for any positive integer μ either (5.11) or (5.14) must hold. If we set $\mu \sim \lambda^{-\left(\frac{n-1}{2}\right)}\sqrt{M}$, then both estimates reduce to

$$
|E| \gtrsim \delta^{n-1} \lambda^{\frac{n+1}{2}} \sqrt{M}.
$$

This is exactly the bound we need to apply Lemma 5.1.1; hence, our restricted weak type estimate follows and we have Bourgain's theorem. □

Using Proposition 2.3.3, we immediately see that Kakeya sets in $\mathbb{R}^n$ have Hausdorff dimension at least $\frac{n+1}{2}$. Of course, we can also use the methods of Section 2.4 to see that Bourgain's theorem implies the estimate $\mathcal{K}(\frac{n+1}{2})$, and so that the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$ is bounded below by $\frac{n+1}{2}$ by Proposition 2.3.1. Whatever route we choose, we are still left with the feeling that this result is clearly far from optimal, at least in light of the Kakeya set conjecure. But before we move onto proving a stronger bound, we will take another look at the bush argument from a different angle. This new perspective will prove to be especially useful when we present Wolff's hairbrush argument in the next chapter.

5.2 Tao's Approach

We introduce the notation $A \lessapprox B$ if one has $A \leq C_{\epsilon}\delta^{-\epsilon}B$ for every $\epsilon > 0$. Thus $A \lessapprox B$ is weaker than $A \lesssim B$, but only by a factor of $\delta^{-\epsilon}$. In the context of the Kakeya conjectures, we see that a Kakeya estimate $A \lessapprox B$ is just as good as $A \lesssim B$ since we can always afford to takea $\delta^{-\epsilon}$ loss. In fact, examining the maximal function conjecture (Conjecture 1.2.2) and Proposition 2.3.1 directly, we see that no finite number of extra $\delta^{-\epsilon}$ factors in our estimates will force us to weaken our conclusions about the norm of the operator or the dimension of Kakeya sets.

One of the attractive features of working with restricted weak type estimates as in Bourgain's argument is that, by definition of the maximal function, we are then led to analyze how dense a given δ-tube is in a set E; this is the substance of (5.5). However, we can simplify matters further by disregarding the arbitrary set E. After all, the geometry of the problem should depend only upon how our δ-tubes interact with each other. Thus, we would like to trim the argument down and simplify the objects we are working with. Tao offers the best path to this slick treatment and we will follow his lecture notes [30] for the remainder of this and the following chapter.

Basically, we want to make use of the duality approach as in Chapter 4. In its present form, this may actually complicate things, so we will first strengthen Lemma 4.1.2 to remove the dependency on our set of directions being maximally δ-separated and on the prescence of the normalizing dual sequence $\{y_k\}$. More precisely, what we are after is the following.

Proposition 5.2.1. *Let $1 < p < \infty$ and let p' be the dual exponent of p. Suppose p has the following property: if $\Omega \subset S^{n-1}$ is any δ-separated set, and if*

$$||\sum_{\omega\in\Omega} \chi_{T_\omega}||_{p'} \lessapprox A \tag{5.15}$$

for any collection of $(1 \times \delta \times \cdots \times \delta)$-tubes $\{T_\omega\}_{\omega\in\Omega}$, then there is a bound

$$||f_\delta^*||_{L^p(S^{n-1})} \lessapprox A||f||_p.$$

This will allow us to streamline the bush argument and concentrate more on the inherent geometrical nature of the problem, rather than getting too bogged down in messy, analytical details. This will prove to be very useful in the next chapter when we present the considerably more complicated hairbrush argument to refine the estimate $\mathcal{K}(\frac{n+1}{2})$ to $\mathcal{K}(\frac{n+2}{2})$. The following version of the bush argument, as well as all of Chapter 6, will liberally exploit the classic pigeonhole principle. We have already used this principle several times, but now it will appear in nearly every reduction we make. It is worth mentioning that this principle will often be disguised as an application of the mean value theorem, its continuous analogue. This simple but powerful tool will aid us in trimming much of the analysis down to a more intuitively palpable level.

To prove Proposition 5.2.1, we will combine two intermediate lemmas.

Lemma 5.2.2. *Let $1 < p < \infty$ and let $\{T_\omega\}_{\omega\in\Omega}$ be any collection of $(1\times\delta\times\cdots\times\delta)$-tubes with orientations from any δ-separated set $\Omega \subset S^{n-1}$. Then a bound of the form*

$$||\sum_{\omega\in\Omega} \chi_{T_\omega}||_{p'} \lessapprox A \cdot \left(\sum_{\omega\in\Omega} |T_\omega|\right)^{\frac{1}{p'}} \tag{5.16}$$

implies a bound

$$||f_\delta^*||_{L^p(S^{n-1})} \lessapprox A||f||_p.$$

Clearly, we can estimate (5.16) as

$$A \cdot \left(\sum_{\omega \in \Omega} |T_\omega| \right)^{\frac{1}{p'}} \sim A \cdot (\delta^{n-1} \# \Omega)^{\frac{1}{p'}} \tag{5.17}$$
$$\lessapprox A,$$

so if (5.16) holds, then (5.15) holds. The amazing thing is though that this implication is reversible.

Lemma 5.2.3. *If* (5.15) *holds, then* (5.16) *holds.*

If we can establish these two lemmas, then we immediately have the proposition by transitivity. We begin with the simpler of the two.

Proof of Lemma 5.2.2. We will prove this lemma by appealing to Wolff's duality argument (Lemma 4.1.2) directly. Let $\Omega \subset S^{n-1}$ be δ-separated and let $\{y_\omega\}_{\omega \in \Omega}$ be such that $\delta^{1-n} = \sum_\omega |y_\omega|^{p'}$. We want to prove that

$$|| \sum_\omega y_\omega \chi_{T_\omega} ||_{p'} \lessapprox A$$

if (5.16) holds. By the triangle inequality, it suffices to show that

$$|| \sum_\omega |y_\omega| \chi_{T_\omega} ||_{p'} \lessapprox A,$$

so there is no harm in assuming $y_\omega \geq 0$. With this in place, define

$$\Omega_{-1} = \{\omega \in \Omega : 0 \leq y_\omega < 1\},$$

$$\Omega_j = \{\omega \in \Omega : 2^j \leq y_\omega < 2^{j+1}\},\ j \geq 0.$$

Clearly then $\Omega = \bigcup_j \Omega_j$ and

$$\delta^{1-n} = \sum_\omega y_\omega^{p'} = \sum_{j=-1}^{\infty} \sum_{\omega \in \Omega_j} y_\omega^{p'} \sim \sum_{j=-1}^{\infty} 2^{jp'} \# \Omega_j. \tag{5.18}$$

Now

$$|| \sum_\omega y_\omega \chi_{T_\omega} ||_{p'} \leq \sum_{j=-1}^{\infty} || \sum_{\omega \in \Omega_j} y_\omega \chi_{T_\omega} ||_{p'}, \tag{5.19}$$

but we would like to remove the dependency on the y_ω's from this expression. As such, we let $z_\omega = 2^{-j} y_\omega$ for all $\omega \in \Omega_j$ so that $z_\omega \sim 1$. Then (5.19) becomes

$$\begin{aligned}\sum_j 2^j || \sum_{\Omega_j} z_\omega \chi_{T_\omega} ||_{p'} &\lesssim \sum_j 2^j || \sum_{\Omega_j} \chi_{T_\omega} ||_{p'} \\ &\lesssim \sum_j 2^j A \cdot \left(\sum_{\Omega_j} |T_\omega| \right)^{\frac{1}{p'}} \\ &\lesssim A \delta^{\frac{n-1}{p'}} \sum_j 2^j (\#\Omega_j)^{\frac{1}{p'}} \qquad (5.20)\end{aligned}$$

by hypothesis. We apply Hölder to the sum in (5.20) and combine with (5.18) and (5.19) to deduce that

$$\begin{aligned}|| \sum_{\Omega} y_\omega \chi_{T_\omega} ||_{p'} &\lesssim A \delta^{\frac{n-1}{p'}} (\#j)^{\frac{1}{p}} \left(\sum_j 2^{jp'} \#\Omega_j \right)^{\frac{1}{p'}} \\ &\lesssim A \delta^{\frac{n-1}{p'}} \left(\log \frac{1}{\delta} \right)^{\frac{1}{p}} \delta^{\frac{1-n}{p'}} \\ &\lessapprox A.\end{aligned}$$

This calculation holds for any δ-separated set Ω, so in particular if Ω is maximally δ-separated, then the hypothesis of Lemma 4.1.2 holds. Thus, its conclusion follows accordingly, giving us the desired bound. □

We proceed to the proof of the second lemma.

Proof of Lemma 5.2.3. For each integer $N \lesssim \delta^{1-n}$, let $c(N)$ denote the smallest constant such that

$$|| \sum_{\omega \in \Omega} \chi_{T_\omega} ||_{p'} \lesssim c(N) \qquad (5.21)$$

holds for all δ-separated sets Ω with $\#\Omega \leq N$. Since we are assuming (5.15) holds, we know that $c(N) \lessapprox 1$; in light of (5.17), we need to improve this estimate to

$$c(N) \lessapprox (\delta^{n-1} N)^{\frac{1}{p'}}. \qquad (5.22)$$

We will use a probability trick to derive this estimate from a recursive inequality.

Suppose that $N \ll \delta^{1-n}$. By definition, we can find a δ-separated set Ω such that $\#\Omega = N$ and

$$|| \sum_{\omega \in \Omega} \chi_{T_\omega} ||_{p'} = c(N).$$

Now let U be a rotation in $\mathbb{R}^n$ selected uniformly at random from the orthogonal group $O(n)$ and consider the set $U(\Omega)$. It is an easy probability to exercise to show that this set will be disjoint from Ω almost surely. If we define $T_{U(\omega)}$ to be the rotated tube $U(T_\omega)$, then by symmetry we have

$$||\sum_{\omega\in U(\Omega)} \chi_{T_\omega}||_{p'} = c(N).$$

The binomial theroem gives us that $||f+g||_q^q \geq ||f||_q^q + ||g||_q^q$ for any $1 \leq q < \infty$ and nonnegative f, g; therefore,

$$||\sum_{\omega\in\Omega\bigcup U(\Omega)} \chi_{T_\omega}||_{p'} \geq 2^{1/p'}c(N). \tag{5.23}$$

Even though $\Omega \cup U(\Omega)$ is a disjoint union, there is nothing to guarantee that it is δ-separated, something that we would obviously like to require. To do this, we need to ensure that we can find an appropriate rotation U, no matter what our initial set of directions Ω may be, so that our larger set of directions $\Omega \cup U(\Omega)$ can be δ-separated after we remove a very small number of its elements.

To this end, we let $a(\Omega, U(\Omega))$ denote the number of pairs in $\Omega \times U(\Omega)$ with first and second coordinates within δ of each other; i.e.

$$a(\Omega, U(\Omega)) = \#\{(\omega, \omega') \in \Omega \times U(\Omega) : |\omega - \omega'| \leq \delta\}.$$

We can write this more explicitly as

$$a(\Omega, U(\Omega)) = \sum_{\omega\in\Omega}\sum_{\omega'\in\Omega} \chi_{B(0,\delta)}(\omega - U(\omega')).$$

If we let U range over the orthogonal group $O(n)$, then we can use the normalized Haar measure[1] $dO(n)$ to write

$$\int_{O(n)} a(\Omega, U(\Omega))dO(n) = \sum_{\omega\in\Omega}\sum_{\omega'\in\Omega} f(\omega, \omega') \tag{5.24}$$

where

$$f(\omega, \omega') = \int_{O(n)} \chi_{B(0,\delta)}(\omega - U(\omega'))dO(n).$$

[1]See Conway [10], pp. 155-56 for the necessary background on the Haar measure of a topological group.

Since we are integrating over the entire group $O(n)$, symmetry considerations tell us that f is independent of the choice of ω'. Thus we can write

$$f(\omega,\omega') = \frac{1}{|S^{n-1}|}\int_{O(n)}\int_{S^{n-1}} \chi_{B(0,\delta)}(\omega - U(\omega'))d\omega' dO(n).$$

If we examine the inner integral only, then ω and U are fixed, so

$$\begin{aligned}\int_{S^{n-1}} \chi_{B(0,\delta)}(\omega - U(\omega'))d\omega' &= \int_{S^{n-1}} \chi_{B(0,\delta)}(\omega - e)de \\ &= |\{e \in S^{n-1} : |\omega - e| \leq \delta\}| \\ &\sim \delta^{n-1},\end{aligned}$$

the surface measure of the ball $B(0,\delta)$. So since $\int_{O(n)} dO(n) = 1$ by definition, we see that $f(\omega,\omega') \sim \delta^{n-1}$ uniformly; thus, (5.24) becomes

$$\int_{O(n)} a(\Omega, U(\Omega))dO(n) \sim \delta^{n-1}N^2,$$

since $\#(\Omega \times U(\Omega)) = N^2$ almost surely.

What this means is that we can always find a rotation U such that $a(\Omega,U(\Omega)) \lessapprox \delta^{n-1}N^2$ just by pigeonholing (or applying the mean value theorem). This is what we are after. Fix one of these rotations U now and partition the set $\Omega \cup U(\Omega)$ into δ-separated pieces. To do this, for any $\Gamma \subseteq S^{n-1}$, let

$$A_\Gamma = \{e \in \Gamma : |e-\omega| \leq \delta \text{ for some } \omega \in \Omega \cup U(\Omega)\}.$$

Now Ω is δ-separated, so $U(\Omega)$ must be also and thus so are the sets A_Ω and $A_{U(\Omega)}$. So we can partition $\Omega \cup U(\Omega)$ into δ-separated parts where $\#A_\Omega = O(\delta^{n-1}N^2)$, $\#A_{U(\Omega)} = O(\delta^{n-1}N^2)$, and $\#(\Omega \cup U(\Omega) \setminus (A_\Omega \cup A_{U(\Omega)})) \leq 2N$, i.e. two small δ-separated sets and one large. Using the definition of $c(N)$ in (5.21) and the triangle inequality, we thus have that

$$\|\sum_{\omega\in\Omega\bigcup U(\Omega)} \chi_{T_\omega}\|_{p'} \leq c(2N) + 2c(C\delta^{n-1}N^2)$$

for some constant C. Combining this with the lower bound in (5.23), we get the inequality

$$c(N) \leq 2^{-1/p'}c(2N) + 2c(C\delta^{n-1}N^2). \qquad (5.25)$$

Now we want to establish (5.22), or equivalently, $c(N)(\delta^{1-n}N^{-1})^{1/p'} \lessapprox 1$. Regardless of our choice of N, we see that there exists some $k \geq 0$ such that $\delta^{1-n}N^{-1} \sim 2^k$; so $N \sim 2^{-k}\delta^{1-n}$. Thus it will suffice to require

$$c_k \sim 2^{k/p'}c(2^{-k}\delta^{1-n}) \lessapprox 1 \qquad (5.26)$$

for all $k \geq 0$. Notice that if $k \gtrsim \log(1/\delta^{n-1})$, then $2^k \gtrsim \delta^{1-n}$ and consequently $N \lesssim 1$, a contradiction. Thus it is enough to prove (5.26) for all $0 \leq k \lesssim \log(1/\delta)$. Multiplying through by the factor $2^{k/p'}$ in (5.25), we see that what we have to work with is the recursive inequality

$$c_k \leq c_{k-1} + C2^{-k/p'} c_{2k} \tag{5.27}$$

for some constant C. Notice that if we did not have the second summand in (5.27) (the error term), then we could conclude immediately that c_k is a nonincreasing sequence in k and thus it attains its maximum at c_0. But $c_0 = c(\delta^{1-n}) \lessapprox 1$ by hypothesis, equation (5.15), so $c_k \lessapprox 1$ for all k. Looking back at (5.26), we see that this would imply that $c(N)(\delta^{1-n}N^{-1})^{1/p'} \lessapprox 1$ for all $N \leq \delta^{1-n}$; this is exactly (5.16), and would verify our lemma.

Unfortunately, we cannot just drop the error term from (5.27) altogether; nevertheless, we would like to assert that c_k behaves as well as a nonincreasing sequence and apply the same subsequent reasoning to conclude our lemma. To do so, we will have to introduce a new sequence d_k defined by

$$d_k = c_k(1 + C_0 2^{-k/p'}),$$

for some sufficiently large C_0 to be chosen later. We claim that if $A > C$, then

$$d_k < d_{k-1} + C2^{-k/p'}(A(d_{2k} - d_k) + (d_{k-1} - d_k)) \tag{5.28}$$

for all $k \geq k_0$, where k_0 is some fixed constant. To verify this claim, multiply (5.27) through by a factor of $(1 + C_0 2^{-k/p'})$ to find that

$$\begin{aligned} d_k &\leq c_{k-1}(1 + C_0 2^{-k/p'}) + C2^{-k/p'} c_{2k}(1 + C_0 2^{-k/p'}) \\ &= d_{k-1}\left(\frac{1 + C_0 2^{-k/p'}}{1 + C_0 2^{-(k-1)/p'}}\right) + C2^{-k/p'} d_{2k}\left(\frac{1 + C_0 2^{-k/p'}}{1 + C_0 2^{-2k/p'}}\right). \end{aligned} \tag{5.29}$$

Subtracting (5.28) from (5.29) and rearranging, we see that we aim to show

$$(A+1)2^{-k/p'} d_k < d_{k-1}\left(1 + 2^{-k/p'} - \frac{1 + C_0 2^{-k/p'}}{1 + C_0 2^{-(k-1)/p'}}\right) + d_{2k} 2^{-k/p'}\left(A - \frac{1 + C_0 2^{-k/p'}}{1 + C_0 2^{-2k/p'}}\right).$$

We substitute in for c_k and simplify to arrive at the proposed inequality

$$c_k < c_{k-1}\left[\frac{1 + C_0(2^{1/p'} - 1) + C_0 2^{-(k-1)/p'}}{(A+1)(1 + C_0 2^{-k/p'})}\right] + C2^{-k/p'} c_{2k}\left[\frac{(A-C)2^{k/p'} - CC_0 + AC_0 2^{-k/p'}}{C(A+1)(1 + C_0 2^{-k/p'})}\right].$$

Since we know (5.27) holds, it is enough to show that each of the bracketed terms above are greater than 1. For the first coefficient, notice for all large k we have

$$\frac{1 + C_0(2^{1/p'} - 1) + C_0 2^{-(k-1)/p'}}{(A+1)(1 + C_0 2^{-k/p'})} > \frac{1 + C_0(2^{1/p'} - 1)}{2(A+1)}$$

and that this can be made greater than 1 just by chosing C_0 large compared to A (recall that A was already fixed to be larger than C). For the second coefficient, we have

$$\frac{(A-C)2^{k/p'} - CC_0 + AC_0 2^{-k/p'}}{C(A+1)(1+C_0 2^{-k/p'})} > \frac{(A-C)2_C^{k/p'} C_0}{2C(A+1)}$$

if k is large enough. Since $A > C$, this can again be made greater than 1 by choosing k appropriately large. Evidently then, we choose a k_0 so that both bounds hold for all $k \geq k_0$.

Thus, the recursive inequality (5.28) holds for all $k \geq k_0$. Let $k_{\max}$ maximize d_k in the range $[k_0, O(\log(1/\delta))]$. Applying (5.28), we have

$$d_{k_{\max}} < d_{k_{\max}-1} + C2^{-k/p'}(A(d_{2k_{\max}} - d_{k_{\max}}) + (d_{k_{\max}-1} - d_{k_{\max}})).$$

Now the second summand must be negative, so we see that $d_{k_{\max}} < d_{k_{\max}-1}$ which forces $k_{\max} = k_0$. Therefore, $c_k < c_{k_0}$ for all $k > k_0$. Notice that we trivially have $c_k \lesssim c_{k_0}$ for all $k < k_0$. So since $c_{k_0} \sim c(2^{-k_0}\delta^{1-n})2^{k_0/p'} \lesssim c(\delta^{1-n}) = c_0 \lessapprox 1$ by hypothesis, we have that $c_k \lessapprox 1$ for all $0 \leq k \lesssim \log(1/\delta)$, completing the proof of Lemma 5.2.3. $\square$

Thus, to prove the estimate $\mathcal{K}(\frac{n+1}{2})$, it suffices to show

$$||\sum_{\omega \in \Omega} \chi_{T_\omega}||_{\frac{n+1}{n-1}} \lessapprox \delta^{-\frac{n-1}{n+1}},$$

where Ω is any δ-separated subset of S^{n-1}, $\{T_\omega\}_{\omega\in\Omega}$ is any collection of δ-tubes with the given orientations. Notice that $p' = \frac{n+1}{n-1}$ and $-\frac{n}{p} + 1 = -\frac{n-1}{n+1}$, so the above expression really is $\mathcal{K}(\frac{n+1}{2})$. Simplifying this, we find that we want to show

$$\int \left(\sum_{\omega \in \Omega} \chi_{T_\omega}(x) \right)^{\frac{n+1}{n-1}} dx \lessapprox \delta^{-1}. \tag{5.30}$$

As mentioned above, we will take great advantage of the pigeonhole principle in the following argument, but we will also employ dyadic decomposition to further discretize the objects of interest. This is another classic analytical tool that will provide us a considerable benefit to simplifying the intuition.

To this end, we proceed to discretize the domain of integration in (5.30). Notice that $\#\Omega = O(\delta^{1-n})$; this follows immediately by applying Fact 5.1.3. Thus, $\sum_{\omega\in\Omega} \chi_{T_\omega}(x) \in [0, O(\delta^{1-n})] \cap \mathbb{Z}$ for any $x \in \mathbb{R}^n$. If we let μ range dyadically from 1 to $O(\delta^{1-n})$ and define the sets

$$E_\mu = \{x : \mu \leq \sum_{\omega\in\Omega} \chi_{T_\omega}(x) < 2\mu\},$$

then we can rewrite the lefthand side of (5.30) as

$$\sum_{\mu} \int_{E_\mu} \left(\sum_{\omega \in \Omega} \chi_{T_\omega}(x) \right)^{\frac{n+1}{n-1}} dx \sim \sum_{\mu} \mu^{\frac{n+1}{n-1}} |E_\mu|.$$

Now the number of μ cannot exceed $\log(\delta^{1-n}) \sim \log 1/\delta$, and since $\log 1/\delta \lessapprox 1$, we see that to prove (5.30) it suffices to show that

$$\mu^{\frac{n+1}{n-1}} |E_\mu| \lessapprox \delta^{-1} \tag{5.31}$$

holds for each μ. Notice that the set E_μ is precisely our bush object; indeed, by definition it is the set of points that lie in μ tubes. Clearly, we need some control over how much of each tube is contained in the set E_μ, i.e. we need to analyze the density of the tubes in this set. This is what the quantity λ did for us before and we will use it again, taking advantage of another dyadic decomposition. Before that though, we make an easy reduction.

Fix μ and suppose $|E_\mu| \le \delta^n$. Then

$$\mu^{\frac{n+1}{n-1}} |E_\mu| \lesssim (\delta^{1-n})^{\frac{n+1}{n-1}} \delta^n = \delta^{-1},$$

and so (5.31) is trivially satisfied. Thus we may assume $|E_\mu| \ge \delta^n$.

With our μ fixed, we use the definition of E_μ to write

$$\mu |E_\mu| \sim \int_{E_\mu} \sum_{\omega \in \Omega} \chi_{T_\omega}(x) dx = \sum_{\omega \in \Omega} |T_\omega \cap E_\mu|.$$

Now the set $|T_\omega \cap E_\mu|$ can range in value from 0 to $|T_\omega|$, so we can discretize this range and for each each dyadic λ between δ^{2n} and $\frac{1}{2}$, define

$$\Omega_\lambda = \{\omega \in \Omega : \lambda |T_\omega| < |T_\omega \cap E_\mu| \le 2\lambda |T_\omega|\};$$

thus, we can write

$$\sum_{\omega \in \Omega} |T_\omega \cap E_\mu| = \sum_{\lambda} \sum_{\omega \in \Omega_\lambda} |T_\omega \cap E_\mu| \; + \sum_{\omega : |T_\omega \cap E_\mu| \le \delta^{2n} |T_\omega|} |T_\omega \cap E_\mu|. \tag{5.32}$$

It is easy to show that the second term can be ignored. First notice that $\sum_\lambda \sum_{\omega \in \Omega_\lambda} |T_\omega \cap E_\mu| \lesssim \mu |E_\mu|$ trivially; then notice that since $\mu \ge 1$, $|E_\mu| \ge \delta^n$, and $\#\Omega \lesssim \delta^{1-n}$, we have

$$\begin{aligned} \sum_{\lambda} \sum_{\omega \in \Omega_\lambda} |T_\omega \cap E_\mu| &\sim \mu |E_\mu| - \sum_{\omega : |T_\omega \cap E_\mu| \le \delta^{2n} |T_\omega|} |T_\omega \cap E_\mu| \\ &\gtrsim \mu |E_\mu| - \delta^{2n} \\ &\gtrsim \frac{1}{2} \mu |E_\mu|. \end{aligned}$$

Consequently, we can simplify (5.32) and just write

$$\mu|E_\mu| \sim \sum_\lambda \sum_{\omega\in\Omega_\lambda} |T_\omega \cap E_\mu|.$$

Plugging in the definition of Ω_λ, we arrive at

$$\mu|E_\mu| \sim \sum_\lambda \lambda\delta^{n-1}\#\Omega_\lambda. \tag{5.33}$$

Now we are going to apply the pigeonhole principle again so that we can analyze just a single term of (5.33). For the sake of precision, we illustrate exactly how to go about doing this.

Define $f : [\delta^{2n}, 1/2] \to \mathbb{R}$ by $f(x) = \#\Omega_\lambda$ if $x = \lambda$ for dyadic λ in the domain and define f by linear interpolation for all other x in the domain. Let η denote counting measure on the set of all λ's. Then

$$\mu|E_\mu| \sim \sum_\lambda \lambda\delta^{n-1}\#\Omega_\lambda = \int_{\delta^{2n}}^{\frac{1}{2}} x\delta^{n-1} f(x)d\eta(x).$$

Now by the mean value theorem, we know that there exists some $x^* \in (\delta^{2n}, 1/2)$ such that

$$\begin{aligned}\mu|E_\mu| &\sim x^*\delta^{n-1} f(x^*) \int_{\delta^{2n}}^{\frac{1}{2}} d\eta(x)\\ &\sim x^*\delta^{n-1} f(x^*)\log(1/\delta)\\ &\lessapprox x^*\delta^{n-1} f(x^*).\end{aligned}$$

Clearly, there exists some λ such that $\lambda < x^* \leq 2\lambda$, so we may write

$$\begin{aligned}\mu|E_\mu| &\lessapprox \lambda\delta^{n-1} f(\lambda)\\ &= \lambda\delta^{n-1}\#\Omega_\lambda\end{aligned}$$

for this particular λ. Since we trivially have that

$$\mu|E_\mu| \sim \sum_{\lambda'} \lambda'\delta^{n-1}\#\Omega_{\lambda'} \geq \lambda\delta^{n-1}\#\Omega_\lambda,$$

we find that, in fact

$$\lambda\delta^{n-1}\#\Omega_\lambda \approx \mu|E_\mu| \tag{5.34}$$

must hold. This procedure may be technical overkill, but it serves as a nice illustration as to how the mean value theorem can make the classic pigeonhole principle analytically precise.

We see that, by (5.34), this particular λ acts as the dominant density of a generic tube T_ω in E_μ. Notice that we have an estimate on this density, namely

$$\lambda \gtrapprox \mu|E_\mu| \tag{5.35}$$

since $\#\Omega_\lambda = O(\delta^{1-n})$. So just as before we have related our multiplicity and our density quantities, μ and λ, and now we can verify (5.31). The remainder of the argument will look very similar to our first pass through the bush argument.

By (5.34), we have that

$$\begin{aligned}\mu|E_\mu| &\approx \sum_{\omega\in\Omega_\lambda} |T_\omega \cap E_\mu| \\ &= \int_{E_\mu} \sum_{\omega\in\Omega_\lambda} \chi_{T_\omega}(x)dx.\end{aligned}$$

So another application of the mean value theorem tells us that there exists at least one high multiplicity, high density point $a \in E_\mu$ for which

$$\sum_{\omega\in\Omega_\lambda} \chi_{T_\omega}(a) \gtrapprox \mu.$$

In other words, this a is contained in $\gtrapprox \mu$ tubes T_ω and each T_ω has density λ inside E_μ.

As before, we trim the intersection $|T_\omega \cap E_\mu|$ by cutting out a ball around a of radius $c_0^{-1}\lambda$ for some sufficiently large constant c_0. As before, we have

$$|T_\omega \cap E_\mu \cap B(a, c_0^{-1}\lambda)^c| \gtrsim \delta^{n-1}\lambda.$$

If we then $c_1\frac{\lambda}{\delta}$-separate our directions $\omega \in \Omega_\lambda$, we know that all intersections as above are disjoint for different choices of $\omega, \omega' \in \Omega_\lambda$. We know that the size of this $c_1\frac{\lambda}{\delta}$-separated subset of Ω_λ, call it Ω_λ^*, has cardinality $\gtrapprox \mu\lambda^{n-1}$, and thus we may estimate

$$\begin{aligned}|E_\mu| &\geq \sum_{\omega\in\Omega_\lambda^*} |T_\omega \cap E_\mu \cap B(a, c_0^{-1}\lambda)^c| \\ &\gtrapprox \mu\lambda^{n-1}\delta^{n-1}\lambda.\end{aligned}$$

Notice that these two inequalities are identical to (5.13) and (5.14). Combining the above with (5.35) and simplifying, we arrive at (5.31) as desired.

Much of this version of the bush argument was similar or identical to our previous rendition, but we were able to develop and employ many new techniques that will prove to be invaluable in the next chapter. Indeed, part of our motivation behind exhibiting a second proof of $\mathcal{K}(\frac{n+1}{2})$ was to take some of the weight off the preliminary analysis required to establish the $\mathcal{K}(\frac{n+2}{2})$, the estimate derived via Wolff's hairbrush argument. As we will see, there are still many reductions to make before we can actually begin to construct this hairbrush argument and analyze it as effectively as we have Bourgain's bush.

Chapter 6

The Hairbrush Argument

Wolff proved a restricted weak type estimate as in Section 5.1 in his original paper [31]. His method was meticulous and often difficult to reconcile with much of the intuitive analysis. Our technical pains of Section 5.2 will be rewarded now by utilizing them, in conjunction with Tao's bilinearization technique, to present a simpler, more intuitively apparent version of Wolff's "hairbrush" argument. This will give us the improved estimate $\mathcal{K}(\frac{n+2}{2})$, affording us an extra $\frac{1}{2}$ dimension for bounding the Hausdorff dimension of Kakeya sets in $\mathbb{R}^n$. This is the best estimate to date for dimensions below nine.

In the bush argument, we quantified and exploited the geometry between a point and a line, analyzing essentially how δ-neighborhoods of lines intersect each other outside a λ-neighborhood of a point. For the hairbrush, as the name suggests, we will quantify and exploit the geometry between a line and another line, analyzing how δ-neighborhoods of lines intersect each other outside a λ-neighborhood of some other fixed line, the "stem" of the hairbrush.

We start with the standard Kakeya maximal operator estimate of interest on the diagonal,

$$||f_\delta^*||_p \lesssim A||f||_p,$$

and use Wolff's duality argument (Lemma 4.1.2) to reduce to an estimate of the form

$$||\sum_\omega y_\omega \chi_{T_\omega}||_{p'} \lesssim A$$

for ω in any maximally δ-separted set of directions $\Omega \subset S^{n-1}$ and any sequence $\{y_\omega\}$ such that $\delta^{n-1}\sum |y_\omega|^{p'} = 1$. Using the triangle inequality to reduce to a positive sequence and normalizing, we apply Tao's factorization result (Proposition 5.2.1) to further reduce the estimate to one of the form

$$||\sum_\omega \chi_{T_\omega}||_{p'} \lessapprox A,$$

where $\omega \in \Omega$ is now just any δ-separated subset of S^{n-1} (not necessarily maximal) and where we probably pick up at least one $\delta^{-\epsilon}$ loss along the way. Substituting our proposed bound, we find that we are aiming to prove an estimate of the form

$$||\sum_{\omega\in\Omega} \chi_{T_\omega}||_{p'} \lessapprox \delta^{-\frac{n}{p}+1}$$

where $p = \frac{n+2}{2}$. Thus, we will prove:

Theorem 6.0.1 (Wolff)**.** *$\mathcal{K}(\frac{n+2}{2})$ holds.*

The hairbrush argument is, in many ways, the logical next step from Bourgain's bush construction. In fact, Wolff's original proof is really quite similar in procedure and in execution to Bourgain's argument. Since we are following Tao's treatment, we will see that our argument is very similar to the bush construction of Section 5.2. It is worth remembering though that both versions of the bush argument relied on analyzing the points in our δ-tubes according to their mulitiplicities and their densities; in addition, both versions quantified and employed the same geometrical facts. This is the case too with Wolff's original hairbrush argument and the one that we present here.

Although the construction of the hairbrush is longer and more complicated than the construction of the bush, it is essential to keep in mind that our general mode of attack is still to analyze points in our δ-tubes according to their multiplicities and their densities (for the hairbrush, there will be two of these for each point) and to exploit certain geometrical facts that quantify how these objects (our δ-tubes) interact in $\mathbb{R}^n$.

6.1 Preliminary Reductions

We begin with several major reductions that will be critical in simplifying the subsequent analysis. We want to verify that $\mathcal{K}(\frac{n+2}{2})$ holds, so we want to show that

$$||\sum_{\omega\in\Omega} \chi_{T_\omega}||_{p'} \lessapprox \delta^{-\frac{n}{p}+1}$$

for $p = \frac{n+2}{2}$. We will refrain from plugging this value of p into our estimates until absolutely necessary to stress the locations of the importance (and the unimportance) of the choice of p.

The bilinear approach leads us to consider the square of our estimate which, after raising everything to the power $p'/2$ looks like

$$||\sum_{\omega\in\Omega}\sum_{\omega'\in\Omega} \chi_{T_\omega}\chi_{T_{\omega'}}||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}.$$

Notice that $p'/2 < 1$, so we cannot be as hasty when applying standard tools like the triangle inequality. Nevertheless, the extra care needed in this bilinear setting will be rewarded by the fact that a new variable becomes available for manipulation, namely the angular separation $|\omega - \omega'|$ between the two directions ω and ω'. To take advantage of this variable, we split the summation at a dyadic scale as

$$\sum_{\omega\in\Omega}\sum_{\omega'\in\Omega} = \sum_{k=0}^{\log(1/\delta)} \sum_{\omega,\omega':|\omega-\omega'|\sim 2^{-k}} + \sum_{\omega,\omega':\omega=\omega'} .$$

Now the triangle inequality does not hold for exponents $r < 1$; however, we can use the pseudo-triangle inequality

$$||\sum_{i=1}^{m} f_i||_r \leq \left(\sum_{i=1}^{m} ||f_i||_r^r\right)^{1/r} .$$

This inequality comes from the fact that $(a+b)^r \leq a^r + b^r$ when $r < 1$, another consequence of the binomial theorem. Applying this fact to the split summation, we find that we would like to show

$$\sum_{k=0}^{\log(1/\delta)} ||\sum_{\omega,\omega':|\omega-\omega'|\sim 2^{-k}} \chi_{T_\omega}\chi_{T_{\omega'}}||_{p'/2}^{p'/2} + ||\sum_{\omega,\omega':\omega=\omega'} \chi_{T_\omega}\chi_{T_{\omega'}}||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}. \tag{6.1}$$

We can easily absorb the diagonal term into the righthand side of (6.1). Just notice that

$$\begin{aligned} ||\sum_{\omega,\omega':\omega=\omega'} \chi_{T_\omega}\chi_{T_{\omega'}}||_{p'/2}^{p'/2} &= \int \left(\sum_{\omega} \chi_{T_\omega}(x)\right)^{p'/2} dx \\ &\leq \sum_{\omega} \int \chi_{T_\omega} \\ &\sim \delta^{n-1} \cdot \#\Omega \\ &\lesssim 1, \end{aligned}$$

and $1 \lesssim \delta^{-\epsilon}$; thus disregarding the diagonal term incurs at most a $\delta^{-\epsilon}$ loss. Now the number of k in (6.1) is $\log(1/\delta) \lessapprox 1$, so to establish (6.1) we only need to show that

$$||\sum_{\omega,\omega':|\omega-\omega'|\sim 2^{-k}} \chi_{T_\omega}\chi_{T_{\omega'}}||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)} \tag{6.2}$$

holds for each k.

Fix k. We are going to reduce our set of directions to a dyadic scale. Recall that all $\omega \in \Omega$ lie within $\frac{1}{10}$ of the vertical e_n. We would like to remove the dependency on k in (6.2). Since we are proving things for any fixed k, it seems reasonable that there should be some type of rescaling argument that allows us to prove (6.2) for only, say, $k = 0$. Now if we 2^{-k}-separate S^{n-1}, then this is just like covering S^{n-1} by $\sim 2^{(n-1)k}$ caps, each of diameter $\sim 2^{-k}$ (as in Fact 5.1.3). Clearly then, we have covered Ω by at most $\sim 2^{(n-1)k}$ caps of diameter $\sim 2^{-k}$. We can choose these caps in such a way so that for each ω, ω' with $|\omega - \omega'| \sim 2^{-k}$, we can find a cap C which contains them both. Thus we may split the sum in (6.2) over these caps and apply the pseudo-triangle inequality again to reduce to

$$\sum_C || \sum_{\omega,\omega' \in C\cap\Omega : |\omega-\omega'|\sim 2^{-k}} \chi_{T_\omega}\chi_{T_{\omega'}} ||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}.$$

Since there are $\sim 2^{(n-1)k}$ caps, we evidently have to prove that

$$|| \sum_{\omega,\omega' \in C\cap\Omega : |\omega-\omega'|\sim 2^{-k}} \chi_{T_\omega}\chi_{T_{\omega'}} ||_{p'/2}^{p'/2} \lessapprox 2^{-(n-1)k}\delta^{n-p'(n-1)} \tag{6.3}$$

holds for each cap C.

Now we want to show that if we have (6.3) for $k = 0$, then we have it for all k. The purpose of introducing the caps and analyzing the sum over an individual cap rather than the entire set Ω will become clear from the following argument.

Assume that (6.3) holds for $k = 0$ and then fix a new $k > 0$. We also assume that we have rotated the cap C to be centered at e_n; this is harmless (i.e. a unitary transformation) and will simplify the argument. Define the transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ by

$$L(2^k\underline{x}, x_n)$$

and the corresponding stretching operator S by

$$Sf(x) = f(L^{-1}x).$$

Notice that $||Sf||_q = 2^{\frac{(n-1)k}{q}}||f||_q$, so (6.3) becomes

$$||S(\sum_{\omega,\omega' \in C\cap\Omega : |\omega-\omega'|\sim 2^{-k}} \chi_{T_\omega}\chi_{T_{\omega'}})||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}.$$

To get this expression into a more tractable form, we write it as

$$|| \sum_{\omega,\omega' \in C\cap\Omega : |\omega-\omega'|\sim 2^{-k}} \chi_{L(T_\omega)}\chi_{L(T_{\omega'})} ||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}. \tag{6.4}$$

Notice that the object $L(T_\omega)$ is not a δ-tube; indeed, if k is very large than it is very far from it. Yet we claim that (6.4) holds by hypothesis. To verify this, we will need to establish another containment lemma with some extra information about the separation of the transformed directions.

Lemma 6.1.1. *Suppose $|\omega - e_n| \lesssim 2^{-k}$. Then there exists a constant C dependent only on n such that*

$$L(T_\omega(a)) \subset C \cdot T^{2^k\delta}_{L(\omega)}(b)$$

where $b = L(a)$, $a \in \mathbb{R}^n$. If in addition, we have ω' such that $|\omega - \omega'| \sim 2^{-k}$, then we have $|L(\omega) - L(\omega')| \sim 1$.

Proof. Pick any $x \in T_\omega(a)$. As in the proof of Lemma 4.1.1, we know that x lies on some unit line segment, call it I, that is parallel to ω and whose midpoint lies within δ of a in the *unique* hyperplane that contains a and has normal vector ω. This is to be absolutely precise, but we can weaken this condition to make things easier. Pick any point $m \in B(0, \delta)$ and let I_m be the unique unit line segment oriented in the ω direction with $a + m$ as midpoint. Then notice that the union of all such line segments must contain the tube $T_\omega(a)$ (in fact, this union will contain more); i.e.

$$T_\omega(a) \subset \bigcup_{m \in B(0,\delta)} I_m. \tag{6.5}$$

The point of this is that we are going to parameterize an arbitrary point in the tube $T_\omega(a)$ according to the line segment I_m that it sits upon.

Now $x \in T_\omega(a)$, so by (6.5), there exists some $m \in B(0, \delta)$ such that we can represent x as follows:

$$x = a + m + t_x\omega, \tag{6.6}$$

where $|t_x| \leq \frac{1}{2}$. Recall that $L(\underline{x}, x_n) = (2^k\underline{x}, x_n)$. Applying L to the representation (6.6), we have

$$L(x) = L(a) + L(m) + t_x L(\omega). \tag{6.7}$$

Since $x \in T_\omega(a)$ is arbitrary, to prove the lemma we have to check that there exists some constant C such that

$$|(L(x) - b) \cdot L(\omega)| \leq C \tag{6.8}$$

and

$$|(L(x) - b) \cdot v| \leq C \cdot 2^k\delta, \tag{6.9}$$

where $v \in L(\omega)^\perp$ is a unit vector.

Via the standard projection formula, we see that $\text{proj}_{e_n}\omega = \cos|\omega - e_n|$; thus,

$$\begin{aligned} |\text{proj}_{e_n^\perp}\omega|^2 &= |\omega|^2 - |\text{proj}_{e_n}\omega|^2 \\ &= 1 - \cos^2|\omega - e_n| \\ &= \sin^2|\omega - e_n| \\ &\lesssim 2^{-2k} \end{aligned}$$

since $|\omega - e_n| \lesssim 2^{-k}$ by hypothesis. Consequently, we have

$$\begin{aligned} |L(\omega) - e_n|^2 &= |(2^k\underline{\omega}, \omega_n - 1)|^2 \\ &= 2^{2k}|\text{proj}_{e_n^\perp}\omega|^2 + |\omega_n - 1|^2 \\ &\lesssim 1. \end{aligned}$$

We use this estimate to derive a bound on $|L(\omega)|$. Indeed, this follows easily now as

$$|L(\omega)| = |L(\omega) - e_n + e_n| \leq |L(\omega) - e_n| + |e_n| \lesssim 1. \tag{6.10}$$

We use this calculation along with the observation that since $m \in B(0, \delta)$, we must have that $|L(m)| \lesssim 2^k\delta$ to show that (6.8) and (6.9) hold.

Now since $L(a) = b$, we use (6.7) to bound the dot product in (6.8) as

$$\begin{aligned} |(L(m) + t_xL(\omega)) \cdot L(\omega)| &\leq |L(m) + t_xL(\omega)| \cdot |L(\omega)| \\ &\lesssim |L(m)| \cdot |L(\omega)| + |L(\omega)|^2 \\ &\lesssim |L(m)| + 1 \\ &\lesssim \delta 2^k + 1 \\ &\lesssim 1. \end{aligned}$$

Thus (6.8) holds. Now for (6.9): pick any unit vector $v \in L(\omega)^\perp$ and notice that $|L(\omega) \cdot v| = 0$ by definition. Consequently, we have

$$\begin{aligned} |(L(m) + t_xL(\omega)) \cdot v| &= |L(m) \cdot v| \\ &\leq |L(m)| \\ &\lesssim 2^k\delta. \end{aligned}$$

This proves the containment part of the lemma. For the separation part, suppose we have a direction ω' such that $|\omega - \omega'| \sim 2^{-k}$. Then clearly $|L(\omega) - L(\omega')| = |L(\omega - \omega')| \leq 2^k|\omega - \omega'| \sim 1$. To establish the opposite inequality, we first note that since $|\omega - \omega'| \sim 2^{-k}$, by the pigeonhole principle there must exist some i, $1 \leq i \leq n$, such that $|\omega_i - \omega_i'| \gtrsim 2^{-k}$.

Suppose there exists some j, $1 \leq j \leq n-1$ such that $|\omega_j - \omega'_j| \gtrsim 2^{-k}$. Then

$$\begin{aligned} |L(\omega) - L(\omega')| &\gtrsim 2^k |\underline{\omega} - \underline{\omega'}| \\ &\gtrsim 2^k |\omega_j - \omega'_j| \\ &\gtrsim 1, \end{aligned}$$

so we are done. If no such j exists, then we have only that $|\omega_n - \omega'_n| \gtrsim 2^{-k}$. We claim that this is not possible. For if $|\omega_n - \omega'_n| \gtrsim 2^{-k}$, then this would imply that ω' is essentially the same as e_n and consequently that there exists some j, $1 \leq j \leq n-1$, such that $|\omega_j - \omega'_j| \gtrsim 2^{-k}$.

To see why this is true, notice that $|\omega - \omega'| \sim 2^{-k}$ and $|\omega - e_n| \lesssim 2^{-k}$ by hypothesis, and this implies that $|\omega' - e_n| \lesssim 2^{-k}$ by the triangle inequality. So since $|\omega - \omega'| \sim 2^{-k}$ we have that

$$\omega' \in C_{e_n}^{2^{-k}} \cap \partial C_{\omega}^{2^{-k}}, \tag{6.11}$$

where C_a^b denotes the cap centered at $a \in S^{n-1}$ of radius $\sim b$ and ∂C denotes the boundary of the cap C. We are supposing that $|\omega_n - \omega'_n| \gtrsim 2^{-k}$, so we also have that

$$\begin{aligned} 2^{-k} &\lesssim |\omega_n - \omega'_n| \\ &\leq |\omega_n - e_n| + |\omega'_n - e_n| \\ &\lesssim 2 \cdot 2^{-k}, \end{aligned}$$

and thus $|\omega - e_n| \sim 2^{-k}$ or $|\omega' - e_n| \sim 2^{-k}$. Without loss of generality, we assume the former holds. Thus we have $\omega \in \partial C_{e_n}^{2^{-k}} \cap \partial C_{\omega'}^{2^{-k}}$. Combining this with (6.11) and the fact that $|\omega_j - \omega'_j| \not\gtrsim 2^{-k}$ for all $1 \leq j \leq n-1$, we see that we must have $\omega' \approx e_n$.

Up to an adjustment of the implicit constants in the following inequalities, we may now take $\omega' = e_n$. Thus, $|\underline{\omega}|^2 = \sin^2 |\omega - \omega'| \sim 2^{-2k}$ and so there must exist some j, $1 \leq j \leq n-1$, such that $|\omega_j - \omega'_j| \sim 2^{-k}$ by the pigeonhole principle. □

Using Lemma 6.1.1, we have

$$|| \sum_{\omega,\omega' \in C \cap \Omega : |\omega - \omega'| \sim 2^{-k}} \chi_{L(T_\omega)} \chi_{L(T_{\omega'})} ||_{p'/2}^{p'/2} \lesssim || \sum_{L(\omega), L(\omega') \in C_{e_n} \cap L(\Omega) : |L(\omega) - L(\omega')| \sim 1} \chi_{T_{L(\omega)}^{2^k\delta}} \chi_{T_{L(\omega')}^{2^k\delta}} ||_{p'/2}^{p'/2},$$

where C_{e_n} is the cap centered at e_n. But the righthand side here is of the form treated in (6.3), which we assumed to hold; here though we have δ replaced by $2^k\delta$, Ω replaced by $L(\Omega)$ and k replaced by 0. Thus we can bound the righthand side by $\lesssim (2^k\delta)^{n-p'(n-1)} = 2^{k(n-p'(n-1))}\delta^{n-p'(n-1)} \leq \delta^{n-p'(n-1)}$ since $p \leq n$ implies $p' \geq \frac{n}{n-1}$; this verifies (6.4).

We have one final reduction to make before entering the estimation phase of the argument. We just proved that it is enough to show that (6.3) holds for $k = 0$, i.e.

$$|| \sum_{\omega,\omega' \in \Omega : |\omega - \omega'| \sim 1} \chi_{T_\omega} \chi_{T_{\omega'}} ||_{p'/2}^{p'/2} \lessapprox \delta^{n-p'(n-1)}. \tag{6.12}$$

Notice that the condition on the summands appears to be different than that in (6.3); however, since $k = 0$, any cap C is essentially just Ω, so we may as well take Ω to be our cap. We want to factor this sum into two so that neither sum depends on the other. To accomplish this, pick some constant $c_0 < |\Omega|$ and cover Ω by caps of this radius. If we do this effeciently, that is, if we c_0-separate the caps with respect to their centers, then we know by Fact 5.1.3 that the number of such caps is a constant N dependent only on the ambient dimension of the space n. Now pick any of these caps - call it C_1 - and define $\Omega_1^{(1)} = C_1 \cap \Omega$. Next, pick any cap - call it C_2 - that has center lying outside C_1', the cap C_1 with rescaled radius $3c_o$. Define $\Omega_2^{(1)} = C_2 \cap \Omega$. Notice that $\text{dist}(\Omega_1^{(1)}, \Omega_2^{(1)}) \sim 1$. Now we repeat this procedure until we have essentially partitioned Ω (it is only an essential partition because we may have double-counted some $\omega \in \Omega$).

The only potential snag in this procedure is that we may reach a point where we cannot define a suitable $\Omega_2^{(j)}$, depending on which caps we chose for our previous pairs. If this is the case, then clearly there are no more than c_n unaccounted caps, c_n some constant depending only on n. But if we chose our c_0 small enough initially, then we can pair these orphan caps off by altering only c_n previous pairings. Thus we can write

$$\Omega = \bigcup_{j=1}^{N} \left(\Omega_1^{(j)} \cup \Omega_2^{(j)} \right),$$

where $\text{dist}(\Omega_1^{(j)}, \Omega_2^{(j)}) \sim 1$ for all j. Thus we can majorize the lefthand side of (6.12) as

$$\begin{aligned} || \sum_{\omega,\omega' \in \Omega : |\omega - \omega'| \sim 1} \chi_{T_\omega} \chi_{T_{\omega'}} ||_{p'/2}^{p'/2} &\leq || \sum_{j=1}^{N} \sum_{(\omega,\omega') \in \Omega_1^{(j)} \times \Omega_2^{(j)}} \chi_{T_\omega} \chi_{T_{\omega'}} ||_{p'/2}^{p'/2} \\ &\leq \sum_{j=1}^{N} || \sum_{\omega \in \Omega_1^{(j)}} \sum_{\omega' \in \Omega_2^{(j)}} \chi_{T_\omega} \chi_{T_{\omega'}} ||_{p'/2}^{p'/2} \end{aligned}$$

by the pseudo-triangle inequality. Since $N \sim 1$, we see that what we aim to prove is that

$$|| \left(\sum_{\omega \in \Omega_1} \chi_{T_\omega} \right) \left(\sum_{\omega' \in \Omega_2} \chi_{T_{\omega'}} \right) ||_{p'/2}^{p'/2} \lessapprox \delta^{n - p'(n-1)}, \tag{6.13}$$

where $\Omega_1, \Omega_2 \subset \Omega$ are separated by ~ 1.

This concludes the reductions phase of the hairbrush argument. We have spent a lot of time and effort trimming the objects down and better quantifying our conditionals so that the subsequent analysis will be as transparent and technically simple as possible. One thing to notice is that we still have not plugged in our particular value for p (recall we are after the estimate $\mathcal{K}(\frac{n+2}{2})$). Indeed, we will continue to refrain from inserting the particular value throughout the estimation phase until absolutely necessary in hope that its exact role in the argument will be better understood.

6.2 Estimation

As in the bush argument, we want to analyze the behavior of the sums in (6.13) according to their multiplicities and densities. Evidently then, we define the set

$$E_{\mu,\mu'} = \{x : \sum_{\omega\in\Omega_1} \chi_{T_\omega}(x) \sim \mu,\ \sum_{\omega'\in\Omega_2} \chi_{T_{\omega'}} \sim \mu'\}.$$

Since Ω is δ-separated, we know that $\#(\Omega) \lesssim \delta^{1-n}$ and so the same holds for any subset of Ω. Consequently, we can let μ, μ' above range dyadically in the interval $[0, O(\delta^{1-n})]$. We can refine this range by rewriting (6.13) as follows:

$$\begin{aligned}\int \left(\sum_{\omega} \chi_{T_\omega}(x) \sum_{\omega'} \chi_{T_{\omega'}}(x)\right)^{p'/2} dx &= \sum_{\mu}\sum_{\mu'} \int_{E_{\mu,\mu'}} \left(\sum_{\omega} \chi_{T_\omega}(x) \sum_{\omega'} \chi_{T_{\omega'}}(x)\right)^{p'/2} dx \\ &\sim \sum_{\mu}\sum_{\mu'} (\mu\mu')^{p'/2} |E_{\mu,\mu'}|,\end{aligned}$$

so clearly there is no harm in considering only μ, μ' in the range $[1, O(\delta^{1-n})]$. Furthermore, since the number of such μ, μ' is on the order of $\log(1/\delta)$, we can ignore the sums, incurring at most a $\delta^{-\epsilon}$ loss. Thus, we aim to prove

$$(\mu\mu')^{p'/2}|E_{\mu,\mu'}| \lessapprox \delta^{n-p'(n-1)}, \tag{6.14}$$

for all dyadic μ, μ' between 1 and $O(\delta^{1-n})$.

Note that if $|E_{\mu,\mu'}|$ is too small, say $\lessapprox \delta^n$, then (6.14) holds trivially since $\mu, \mu' \lesssim \delta^{1-n}$; so it suffices to take $|E_{\mu,\mu'}| \gtrapprox \delta^n$.

Now we can impose our density structure on the problem, extracting only those tubes with a certain density of appropriate multiplicity points, i.e. points in $E_{\mu,\mu'}$. Specifically, fix μ, μ' and define

$$\begin{aligned}\Omega_1^\lambda &= \{\omega \in \Omega_1 : |T_\omega \cap E_{\mu,\mu'}| \sim \lambda |T_\omega|\}, \\ \Omega_2^{\lambda'} &= \{\omega' \in \Omega_2 : |T_{\omega'} \cap E_{\mu,\mu'}| \sim \lambda' |T_{\omega'}|\}.\end{aligned}$$

Notice that if λ is near 0, then the set Ω_1^λ identifies precisely those tubes that have a correspondingly small proportion of μ-multiplicity points. The same holds for λ' and the second set. These sets allow us to keep track of the multiplicity points by quantifying how dense these points are in any given tube.

To handle (6.14), we will estimate in two phases, each time translating the quantities into density statements. The first estimate is easy and will handle a factor of $(\mu\mu')^{1/2}|E_{\mu,\mu'}|$; notice that this factor does not depend on p. The second estimate will require us to construct Wolff's hairbrush object which will be utitlized to bound the remaining factor, $(\mu\mu')^{\frac{p'-1}{2}}$. It is this second estimate that will force us to chose the appropriate value for p and this choice will be forced upon us as a direct result of the geometrical lemma we will use. Again, it is a simple geometric fact that makes and breaks the entire argument.

The first real order of business is to reduce matters to when λ, λ' are both sufficiently large, the high density case. The following lemma will allow us to ignore the three alternatives.

Lemma 6.2.1. *There exist dyadic* $\lambda, \lambda' \in [\delta^{Cn}, 1]$ *such that*

$$\int_{E_{\mu,\mu'}} \sum_{\omega \in \Omega_1^{\lambda}} \sum_{\omega' \in \Omega_2^{\lambda'}} \chi_{T_\omega} \chi_{T_{\omega'}} \gtrapprox \mu\mu' |E_{\mu,\mu'}|, \tag{6.15}$$

where $C = \frac{3n-1}{n}$.

It should be stressed that this lemma will be the crucial first step in deriving both estimates described above. Naturally, the proof is a rather tedious pigeonholing argument and can be skipped if self-evident to the reader.

Proof. By definition of $E_{\mu,\mu'}$ we have

$$\int_{E_{\mu,\mu'}} \sum_{\omega \in \Omega_1} \sum_{\omega' \in \Omega_2} \chi_{T_\omega} \chi_{T_{\omega'}} \sim \mu\mu' |E_{\mu,\mu'}|.$$

We split the sums over the subsets $\Omega_1^{\lambda}, \Omega_2^{\lambda'}$ and sum over all dyadic densities, taking care to separate the high and low density terms. Up to a constant, the above integral becomes

$$\left(\sum_{\lambda \geq \delta^{Cn}} \sum_{\lambda' \geq \delta^{Cn}} + \sum_{\lambda < \delta^{Cn}} \sum_{\lambda' \geq \delta^{Cn}} + \sum_{\lambda \geq \delta^{Cn}} \sum_{\lambda' < \delta^{Cn}} + \sum_{\lambda < \delta^{Cn}} \sum_{\lambda' < \delta^{Cn}} \right) \int_{E_{\mu,\mu'}} \sum_{\omega \in \Omega_1^{\lambda}} \sum_{\omega' \in \Omega_2^{\lambda'}} \chi_{T_\omega} \chi_{T_{\omega'}}.$$

To prove the lemma, it suffices to show that the last three terms above are each bounded by say $\frac{1}{10}\mu\mu'|E_{\mu,\mu'}|$. We can then absorb these terms into the righthand side of (6.15). The remaining term will be a sum over $\lambda, \lambda' < \delta^{Cn}$. The number of such λ, λ' is $\lessapprox 1$, so by a now familiar application of the pigeonhole principle (or mean value theorem), we can conclude that λ, λ' exist so that (6.15) holds.

Consider the first of the three terms from above. Since $\#(\Omega_1^{\lambda}), \#(\Omega_2^{\lambda'}) \lessapprox \delta^{1-n}$ and

$$\int_{E_{\mu,\mu'}} \chi_{T_\omega} \chi_{T_{\omega'}} \leq |T_\omega \cap E_{\mu,\mu'}| \sim \lambda \delta^{n-1}$$

for any $\omega \in \Omega_1^\lambda$, we see that this term is bounded by

$$\sum_{\lambda<\delta^{Cn}} \sum_{\lambda' \geq \delta^{Cn}} \lambda \delta^{1-n}.$$

Now $\lambda' \in [\delta^{Cn}, 1]$ ranges dyadically, so the number of λ' is $\lesssim \log(1/\delta)$. We can now bound the previous by

$$\begin{aligned} \delta^{1-n} \log(1/\delta) \sum_{\lambda<\delta^{Cn}} \lambda \sim \delta^{1-n} \log(1/\delta) \sum_{j=\log(1/\delta^{Cn})}^{\infty} 2^{-j} \\ \lesssim \delta^{1-n} \log(1/\delta) \delta^{Cn}. \end{aligned}$$

By our choice of C, this quantity is no more than $\delta^{2n} \log(1/\delta) \lessapprox \frac{1}{10}\mu\mu'|E_{\mu,\mu'}|$ since we already know that $|E_{\mu,\mu'}| \gtrapprox \delta^n$. This takes care of the first of the three terms and the second is dealt with in the same way by the symmetry of the above argument.

For the final term, it is clear that we can bound it by

$$\delta^{1-n} \sum_{\lambda<\delta^{Cn}} \sum_{\lambda'<\delta^{Cn}} \min(\lambda, \lambda')$$

using the same reasoning as before. We rewrite this sum as

$$\delta^{1-n} \sum_{j=\log(1/\delta^{Cn})}^{\infty} \sum_{j'=\log(1/\delta^{Cn})}^{\infty} \min(2^{-j}, 2^{-j'}).$$

Now for any fixed j, we see that $\min(2^{-j}, 2^{-j'}) \leq 2^{-j}$ for the first $j - \log(1/\delta^{Cn})$ terms of the sum over j'; for the remaining values of j', we simply have $\min(2^{-j}, 2^{-j'}) = 2^{-j'}$. Thus, the inner sum over j' can be dominated by

$$\sum_{j'=j}^{\infty} 2^{-j'} + (j - \log(1/\delta^{Cn}))2^{-j} \lesssim 2^{-j} + j2^{-j} \lesssim j2^{-j}.$$

Consequently, after a shift of the index, the entire expression is bounded by

$$\delta^{1-n}\delta^{Cn} \sum_{j=0}^{\infty} (j + \log(1/\delta^{Cn}))2^{-j} \lesssim \delta^{1-n}\delta^{Cn} \log(1/\delta),$$

which again gives the desired bound by our choice of C and condition on the size of $E_{\mu,\mu'}$. □

With these particular λ, λ' fixed, we proceed to the estimation. As described above, this will consist of two phases where we estimate separate factors of (6.14) in terms of the densities λ, λ'. The first estimate is an almost immediate consequence of the previous lemma.

By construction, we have that

$$\sum_{\omega' \in \Omega_2^{\lambda'}} \chi_{T_{\omega'}}(x) \lesssim \mu'$$

whenever $x \in E_{\mu,\mu'}$. Thus by equation (6.15) of Lemma 6.2.1, we have

$$\mu |E_{\mu,\mu'}| \lessapprox \int_{E_{\mu,\mu'}} \sum_{\omega \in \Omega_1^{\lambda}} \chi_{T_\omega} \sim \lambda \delta^{n-1} \#(\Omega_1^{\lambda}).$$

Since $\#(\Omega_1^{\lambda}) \lesssim \delta^{1-n}$, it follows that $\mu|E_{\mu,\mu'}| \lessapprox \lambda$. Similarly, we have $\mu'|E_{\mu,\mu'}| \lessapprox \lambda'$.

Plugging these estimates into (6.14), we see that we now want to show

$$(\mu\mu')^{\frac{p'-1}{2}} (\lambda\lambda')^{\frac{1}{2}} \lessapprox \delta^{n-p'(n-1)}. \tag{6.16}$$

To deal with this second factor, we will need to construct the hairbrush object and convert it into a quantitative upper bound for the product of the multiplicities, $\mu\mu'$. We construct this object as follows.

By a simple shuffling of equation (6.15), we have that

$$\sum_{\omega' \in \Omega_2^{\lambda'}} \int_{T_{\omega'}} \sum_{\omega \in \Omega_1^{\lambda}} \chi_{T_\omega} \gtrapprox \mu\mu' |E_{\mu,\mu'}|.$$

Since $\#(\Omega_2^{\lambda'}) \lesssim \delta^{1-n}$, there must exist some ω' such that

$$\int_{T_{\omega'}} \sum_{\omega \in \Omega_1^{\lambda}} \chi_{T_\omega} \gtrapprox \delta^{n-1} \mu\mu' |E_{\mu,\mu'}|.$$

Fix this ω' and rewrite the above as

$$\sum_{\omega \in \Omega_1^{\lambda}} |T_\omega \cap T_{\omega'}| \gtrapprox \delta^{n-1} \mu\mu' |E_{\mu,\mu'}|.$$

Recall that our preliminary reductions assured us that taking $|\omega - \omega'| \sim 1$ is sufficient. So now we may use Córdoba's estimate on the size of the intersection of our tubes (Lemma 4.1.1) to assert that $|T_\omega \cap T_{\omega'}| \lesssim \delta^n$. Plugging this into the above and simplifying, we have

$$\#\{\omega \in \Omega_1^{\lambda} : T_\omega \cap T_{\omega'} \neq \emptyset\} \gtrapprox \delta^{-1} \mu\mu' |E_{\mu,\mu'}|. \tag{6.17}$$

This completes the construction of the hairbrush. We think of the fixed tube $T_{\omega'}$ as the stem of this brush and imagine that each T_ω is a bristle extending out from this stem. These bristles are nicely separated from the stem (we can think that none of the bristles are too matted down) by the angular separation condition on ω and ω'. Each of these bristles contains a high proportion, λ, of multiplicity points (points in $E_{\mu,\mu'}$) and the above estimate assures us that there are many such bristles - at least $\delta^{-1}\mu\mu'|E_{\mu,\mu'}|$ in fact. En route to equation (6.16), we would like to convert the lefthand side of (6.17) into one involving $|E_{\mu,\mu'}|$ and appropriate powers of λ and δ. This procedure, as well as the form of (6.17), should seem very familiar. Clearly, we need to make some kind of combinatorial (and geometric) argument to count the number of tubes T_ω with orientations in Ω_1^λ that intersect the stem of our hairbrush. Once we have accomplished this, our analysis should be essentially complete.

Let $\mathbf{T}$ denote the collection of tubes $\{T_\omega : \omega \in \Omega_1^\lambda,\ T_\omega \cap T_{\omega'} \neq \emptyset\}$; thus

$$\#(\mathbf{T}) \gtrapprox \delta^{-1}\mu\mu'|E_{\mu,\mu'}|.$$

By definition, for each $T_\omega \in \mathbf{T}$, we have

$$\int_{E_{\mu,\mu'}} \chi_{T_\omega} \sim \lambda\delta^{n-1}.$$

As before, it will be convenient to cut out a small portion of the integral. In natural analogy with the bush argument, we remove a cylindrical neighborhood of the stem tube $T_{\omega'}$ to utilize the near disjointness of the remaining portions of the bristle tubes.

To make this precise, define $\Gamma = \{x : \mathrm{dist}(x, T_{\omega'}) > C^{-1}\lambda\}$ for some sufficiently large constant C. Now Γ^c is a cylindrical neighborhood of $T_{\omega'}$ and we can imagine covering this neighborhood by an essentially disjoint collection of δ-tubes all parallel to $T_{\omega'}$. A moment's thought reveals that there are at most $\lesssim \frac{\lambda}{C\delta}$ such tubes.

Consider the portion of the above integral contained in this neighborhood of $T_{\omega'}$. Applying the Córdoba estimate to the intersection of each tube in the cover with T_ω, we see that

$$\int_{E_{\mu,\mu'}} \chi_{T_\omega \cap \Gamma^c} \lesssim \frac{\lambda}{C\delta}\delta^n < \frac{1}{2}\lambda\delta^{n-1}$$

for C chosen sufficiently large. Thus we have

$$\int_{E_{\mu,\mu'}} \chi_{T_\omega \cap \Gamma} \gtrsim \lambda\delta^{n-1}$$

and summing this estimate over all $T_\omega \in \mathbf{T}$, we arrive at

$$\int_{E_{\mu,\mu'}} \sum_{T_\omega \in \mathbf{T}} \chi_{T_\omega \cap \Gamma} \gtrsim \lambda\delta^{n-1}\#(\mathbf{T}).$$

To further exploit the geometry of the situation, we need to analyze the intersection properties of a pair of tubes in $\mathbf{T}$; thus we apply Cauchy-Schwarz above and rewrite the estimate as

$$||\sum_{T_\omega \in \mathbf{T}} \chi_{T_\omega \cap \Gamma}||_2 \gtrsim \lambda \delta^{n-1} \#(\mathbf{T}) |E_{\mu,\mu'}|^{-1/2}. \tag{6.18}$$

We will show that

$$||\sum_{T_\omega \in \mathbf{T}} \chi_{T_\omega \cap \Gamma}||_2 \lessapprox (\#(\mathbf{T}) \lambda^{2-n} \delta^{n-1})^{1/2} \tag{6.19}$$

and so attain an upper bound for $\#(\mathbf{T})$ after an obvious simplification. Casting a mindful eye back to equation (6.17), we see that an upper bound on $\#(\mathbf{T})$ will give us an estimate in terms of all the quantities that we can control.

Square the estimate in (6.19) and apply Fubini:

$$\sum_{T_{\omega_1} \in \mathbf{T}} \sum_{T_{\omega_2} \in \mathbf{T}} |T_{\omega_1} \cap T_{\omega_2} \cap \Gamma| \lessapprox \#(\mathbf{T}) \lambda^{2-n} \delta^{n-1}.$$

Clearly it suffices to show that

$$\sum_{T_{\omega_2} \in \mathbf{T} : T_{\omega_1} \cap T_{\omega_2} \cap \Gamma \neq \emptyset} |T_{\omega_1} \cap T_{\omega_2} \cap \Gamma| \lessapprox \lambda^{2-n} \delta^{n-1}$$

for all $T_{\omega_1} \in \mathbf{T}$.

Fix T_{ω_1} and split the sum over the angle of separation between ω_1 and ω_2; thus the above estimate becomes

$$\sum_{k=0}^{\log(1/\delta)} \sum_{T_{\omega_2} \in \mathbf{T} : \theta(\omega_1, \omega_2) \sim 2^{-k},\ T_{\omega_1} \cap T_{\omega_2} \cap \Gamma \neq \emptyset} |T_{\omega_1} \cap T_{\omega_2} \cap \Gamma| \lessapprox \lambda^{2-n} \delta^{n-1}.$$

Notice that we do not need to consider the term where $\omega_1 = \omega_2$ since in this case $|T_{\omega_1} \cap T_{\omega_2} \cap \Gamma| \sim \delta^{n-1} \leq \lambda^{2-n} \delta^{n-1}$ trivially. Now clearly the number of k is ≈ 1, so we may show the estimate for each k individually.

Fix k so that $\theta(\omega_1, \omega_2) \sim 2^{-k}$. We use Córdoba's estimate once again to bound the intersection of T_{ω_1} and T_{ω_2} by $2^k \delta^n$. Thus this same quantity bounds $|T_{\omega_1} \cap T_{\omega_2} \cap \Gamma|$ and so after a rearrangement of the previous inequality, we reduce to showing that

$$\#\{T_{\omega_2} \in \mathbf{T} : \theta(\omega_1, \omega_2) \sim 2^{-k},\ T_{\omega_1} \cap T_{\omega_2} \cap \Gamma \neq \emptyset\} \lesssim 2^{-k} \delta^{-1} \lambda^{2-n}. \tag{6.20}$$

This estimate will follow from our critical geometric lemma, the proof of which we defer to the next section.

Lemma 6.2.2 (Wolff)**.** *Fix a tube $T_{\omega'}$. Suppose T_{ω_1} and T_{ω_2} are two other tubes that both intersect the stem $T_{\omega'}$ at an angle ~ 1. Suppose also that $\theta(\omega_1, \omega_2) \sim 2^{-k}$ for some $0 \leq k \leq \log(1/\delta)$ and that $T_{\omega_1} \cap T_{\omega_2} \cap \Gamma \neq \emptyset$. Consider the unique 2-plane π generated by the directions ω' and ω_1 and translate this plane in $\mathbb{R}^n$ so that it contains the principal axis of $T_{\omega'}$. Let π^* denote the $O(\delta/\lambda)$ neighborhood of this translate of π. Then $T_{\omega_2} \subset \pi^*$.*

Taking this lemma for granted for the moment, we return to verifying equation (6.20). It is clear that the set on the lefthand side can be thought of as a set of directions in S^{n-1} obeying the given criteria. Since $\theta(\omega_1, \omega_2) \sim 2^{-k}$, we immediately have that $\theta(\omega_1, \text{proj}_\pi(\omega_2)) \lesssim 2^{-k}$. Now $\pi^\perp$ is an $(n-2)$-dimensional subspace, the span of $n-2$ linearly independent vectors in $\mathbb{R}^n$; denote these vectors by $v_1, \ldots, v_{n-2}$. Under the hypotheses of the lemma, we have

$$\text{proj}_{\frac{v_i}{||v_i||}}(\omega_2) \lesssim \frac{\delta}{\lambda}$$

for all $1 \leq i \leq n-2$. Thus the set of all directions in (6.20) is contained in a cap of volume $\lesssim 2^{-k}(\frac{\delta}{\lambda})^{n-2}$. The valid directions in (6.20) are automatically δ-separated, so we may δ-separate the cap to find that the *number* of such directions cannot exceed $\lesssim 2^{-k}\delta^{-1}\lambda^{2-n}$. Thus we have (6.20).

Now we retrace our steps and put the pieces together. Estimate (6.20) gives an upper bound on (6.18):

$$\lambda\delta^{n-1}\#(\mathbf{T})|E_{\mu,\mu'}|^{-1/2} \lessapprox (\#(\mathbf{T})\lambda^{2-n}\delta^{n-1})^{1/2}.$$

Simplifying this expression and solving for $\#(\mathbf{T})$, we find that

$$\#(\mathbf{T}) \lessapprox \lambda^{-n}\delta^{1-n}|E_{\mu,\mu'}|.$$

Combining this with (6.17), we arrive at the bound

$$\mu\mu' \lessapprox \lambda^{-n}\delta^{2-n}.$$

By the symmetry of the argument, we also have that

$$\mu\mu' \lessapprox (\lambda')^{-n}\delta^{2-n}.$$

Taking the geometric mean of these two estimates, we deduce that

$$\mu\mu' \lessapprox (\lambda\lambda')^{-\frac{n}{2}}\delta^{2-n}.$$

Recall that we were ultimately after the estimate in (6.16). Here, we are finally forced to fix the value of p; we set $p = \frac{n+2}{2}$. Applying the above estimate then, (6.16) becomes

$$(\mu\mu')^{\frac{1}{n}}(\lambda\lambda')^{\frac{1}{2}} \lessapprox \delta^{\frac{2-n}{n}} = \delta^{n-p'(n-1)}.$$

Notice that the power of δ is completely determined by the power of the geometric lemma above - nowhere else in the estimation do we accrue anymore than a $\delta^{-\epsilon}$ loss. This should decisively reaffirm the verity that a Kakeya estimate is only as tight as its corresponding geometrical fact.

6.3 The Geometrical Lemma

At its core, this geometrical fact tells us - quantitatively - that if we have two bristles that our close together, then we can contain one in a thickening of the plane generated by the principal axes of the other bristle and the stem itself. This seems intuitive enough, but as with any analysis of the Kakeya problem, we must reinforce our intuition on a firm analytical foundation, else risk oversimplifying the problem's more precarious delicacies.

Lemma 6.3.1 (Wolff)**.** *Fix $\omega' \in \Omega_2^\lambda$ and $\omega_1 \in \Omega_1^\lambda$ such that $T_{\omega'} \cap T_{\omega_1} \neq \emptyset$; notice that by definition, $\theta(\omega', \omega_1) \sim 1$. Pick an $\omega_2 \in \Omega_1^\lambda$ such that $\theta(\omega_1, \omega_2) \sim \varphi$ for $\delta \leq \varphi \lesssim 1$, $T_{\omega'} \cap T_{\omega_2} \neq \emptyset$ and $T_{\omega_1} \cap T_{\omega_2} \cap \Gamma \neq \emptyset$ where Γ is the complement of a $C^{-1}\lambda$-neighborhood of the stem tube $T_{\omega'}$. Let π be the 2-plane determined by the directions ω' and ω_1 and translate this plane in $\mathbb{R}^n$ so that it contains the principal axis of $T_{\omega'}$. Let π^* denote the $O(\delta/\lambda)$ neighborhood of this translate of π. Then $T_{\omega_2} \subset \pi^*$.*

Proof. We begin stretching the tubes $T_{\omega'}$ and T_{ω_1} so that the directions corresponding to the new orientations are orthogonal. This stretch is harmless since these tubes were already separated by ~ 1 and even though these objects are no longer tubes exactly, they still contain and are contained in an innocuous rescaling of the original tubes, as in Lemma 6.1.1 with $k = 0$. Notice that the same holds for T_{ω_2} and that this tube is now almost orthogonal to the stem $T_{\omega'}$ while we still have that $\theta(\omega_1, \omega_2) \sim \varphi$. Next, we reduce to the case where $T_{\omega'}$ and T_{ω_1} are both centered at the origin, as in the proof of Córdoba's geometrical fact, Lemma 4.1.1. Now we rotate the system so that $\omega' \parallel e_{n-1}$ and $\omega_1 \parallel e_n$. Notice that $\pi(e_{n-1}, e_n) \subseteq \pi_\delta(\omega', \omega_1)$ where $\pi_\delta(\omega', \omega_1)$ is a $\sim \delta$-neighborhood of the 2-plane $\pi(\omega', \omega_1)$. So we see that if we can contain T_{ω_2} in a $\sim \delta/\lambda$-thickening of $\pi(e_{n-1}, e_n)$, then we can contain T_{ω_2} in a $\sim 2\delta/\lambda$-thickening of $\pi(\omega', \omega_1)$. It thus suffices to work with the 2-plane $\pi(e_{n-1}, e_n)$.

Let $x \in T_{\omega_2} \cap \Gamma$ and write the components of x in spherical coordinates so that

$$x_1 = r \sin(\phi_{n-1}) \cdots \sin(\phi_2) \sin(\phi_1)$$

$$x_2 = r \sin(\phi_{n-1}) \cdots \sin(\phi_2) \cos(\phi_1)$$

$$\cdots$$

$$x_{n-1} = r \sin(\phi_{n-1}) \cos(\phi_{n-2})$$

$$x_n = r \cos(\phi_{n-1})$$

where $\phi_1 \in [0, 2\pi)$, $\phi_j \in [0, \pi)$ for all $2 \leq j \leq n-1$. We want to synthesize our conditions on T_{ω_2} with this parametrization. Notice that for any point $x \in T_{\omega_2} \cap \Gamma$ we have $\phi_{n-1} \sim \theta(\omega_2, e_n) \sim \varphi$. Consequently, we see that $\sin(\phi_{n-1}) \sim \sin(\varphi)$ and $\cos(\phi_{n-1}) \sim \cos(\varphi)$.

Now T_{ω_1} and T_{ω_2} intersect in Γ so there must exist some $h \gtrsim \lambda$ and some $a_1, \ldots, a_{n-1} \in B_{n-1}(0, C\delta)$ (the $C\delta$-ball centered at the origin in $\mathbb{R}^{n-1}$) such that

$$(a_1, \ldots, a_{n-1}, h) \in T_{\omega_1} \cap T_{\omega_2} \cap \Gamma.$$

Consider the line segment

$$(a_1, \ldots, a_{n-1}, h) + t(\sin(\varphi)\sin(\phi_{n-2}) \cdots \sin(\phi_1), \ldots, \cos(\varphi))$$

for $-1 \lesssim t \lesssim 1$. Then any line segment of T_{ω_2} parallel to its principal axis is contained in a $\sim 2\delta$-tube with principal axis given by the above line segment. Now the place where T_{ω_2} intersects the stem tube $T_{\omega'}$ is near the point where the above line segment has $x_n = 0$ (more precisely, this intersection happens within $\lesssim \delta$ of $x_n = 0$). The point $x_n = 0$ corresponds to

$$t = \frac{h}{-\cos(\varphi)}.$$

To ensure that T_{ω_2} and the stem $T_{\omega'}$ do indeed intersect here, we should require that $|x_1|, \ldots, |x_{n-2}| \lesssim \delta$. Consider the first inequality, $|x_1| \lesssim \delta$. Since $|a_1| \lesssim \delta$, we require that $|t \sin(\varphi) \cdots \sin(\phi_1)| \lesssim \delta$. Plugging in our requisite value of t, we find that we must have

$$\left|\frac{h}{\cos(\varphi)} \cdot \sin(\varphi) \cdots \sin(\phi_1)\right| \lesssim \delta.$$

Now $-1 \lesssim t \lesssim 1$, so if we can ensure that $|x_1| \lesssim \frac{\delta}{\lambda}$ at $|t| \sim 1$, then we are done. Indeed, if $|t| \sim 1$, then

$$|x_1| \sim |a_1 + \sin(\varphi) \cdots \sin(\phi_1)| \lesssim \delta + |\sin(\varphi) \cdots \sin(\phi_1)|.$$

By the previous inequality, we see that this is bounded by

$$\lesssim \delta + \frac{\delta}{h}\cos(\varphi) \lesssim \frac{\delta}{\lambda}$$

since $h \gtrsim \lambda$ and $\lambda \leq 1$. The same argument shows that $|x_2|, \ldots, |x_{n-2}| \lesssim \frac{\delta}{\lambda}$.

So our line segment lies within a δ/λ-neighborhood of the plane $\pi(e_{n-1}, e_n)$ and since our tube T_{ω_2} was contained in a $\sim 2\delta$-tube with this line segment as its principal axis, we see that T_{ω_2} is contained in a $\sim 2\delta/\lambda$-neighborhood of the same plane. This verfies the lemma. □

This completes our presentation of Wolff's hairbrush argument. We leave this chapter with the knowledge that $\mathcal{K}(\frac{n+2}{2})$ holds and so that Kakeya sets in $\mathbb{R}^n$ must have Hausdorff dimension at least $\frac{n+2}{2}$. To date, this is still the best bound for all dimensions less than 9. There is an argument by Katz and Tao [21] that shows that $\mathcal{K}(\frac{4n+3}{7})$ holds; this improves the maximal function estimate - and so the Hausdorff bound - in higher than 9 dimensions. In fact, using more general combinatorial techniques than what we have described here, they have managed to improve the estimate on the Hausdorff bound beyond the maximal function bound to $(2 - \sqrt{2})(n - 4) + 3$, an upgrade over Wolff's estimate in higher than 5 dimensions. As playing around with different values of n in these bounds shows though, there is still much work to be done. It is likely that a full resolution of the Kakeya problem will require powerful, or perhaps just extremely perceptive, new combinatorial and geometric techniques.

The final chapter will attempt to explore some cousin problems of Kakeya. We will discuss circular and higher dimensional analogues of the classic Kakeya problem, as well as the finite field equivalent - in this setting, we will prove the optimal result, due to Dvir [13]. The final section will endeavor to frame Kakeya in the tapestry of many famous and currently intractable problems in harmonic analysis.

Chapter 7

Variants of the Classical Problem

In this chapter we explore some different Kakeya-type problems. These variants address similar issues of dimensionality of certain sparse sets arising from geometrically rich conditions. We shall see that in some of these settings the problem is much simpler to deal with, but there is still much left that is unknown.

7.1 Circular Kakeya Sets

Although it is suspected that these sets were studied to some extent previously, the 1999 paper by Kolasa and Wolff [23] seems to be the initial systematic and concrete study of what can be considered the circular Kakeya problem. All of the material of the following section and much more is contained in their cited paper. The objective here is the same as before: to obtain certain L^p estimates on a properly defined maximal operator and so deduce a bound on the Hausdorff dimension of circular Kakeya sets in $\mathbb{R}^n$, which we define now.

Definition 7.1.1. *A Borel set $A \subset \mathbb{R}^n$ is a* ***circular Kakeya set*** *if it contains a sphere of every radius.*

The simultaneous publications of Besicovitch and Rado [5] and Kinney [18] exhibited closed circular Kakeya sets in $\mathbb{R}^2$ with Lebesgue measure zero. The construction of Besicovitch and Rado also works in $\mathbb{R}^n$ for $n \geq 3$ as observed by Kolasa and Wolff (again, see their paper for details), and gives us a closed circular Kakeya set in $\mathbb{R}^n$ with Lebesgue measure zero for any $n \geq 3$.

As with the classical problem, we can ask whether circular Kakeya sets must have full Hausdorff dimension. In the context of what has been studied in this essay so far, it is quite surprising that this question can be easily answered in the affirmative for all dimensions $n \geq 3$, but that in $\mathbb{R}^2$, the question remains open. We have the two main theorems of Kolasa and Wolff.

Theorem 7.1.2. *In $\mathbb{R}^n$, $n \geq 3$, circular Kakeya sets have full Hausdorff dimension n.*

Theorem 7.1.3. *In $\mathbb{R}^2$, circular Kakeya sets have Hausdorff dimension at least $\frac{11}{6}$.*

We will prove Theorem 7.1.2 in detail below and the proof will bear a striking resemblance to the proof of Córdoba's L^2 estimate for the classical Kakeya problem. Once our definitions are properly made and our geometrical lemma established, the technical argument will read identically.

Theorem 7.1.3 is more involved and we once again refer the reader to [23] for a proper treatment. It is ironic perhaps that in the setting of circular Kakeya sets, we have an optimal result for the exact range of n that we do not have an optimal result for in the classic problem, and vice-versa. [Why is this? What is behind this? Speculate or say nothing? I haven't been able to think of a reasonable way to suggest why this should be the case.]

We define a natural maximal operator for our new problem as follows: fix $0 < \delta \ll 1$ and for $x \in \mathbb{R}^n$, $r \in [\frac{1}{2}, 2]$, define

$$C(a,r) = \{x \in \mathbb{R}^n : |x - a| = r\}$$

$$C_\delta(a,r) = \{x \in \mathbb{R}^n : r - \delta < |x - a| < r + \delta\}.$$

Evidently then, $C_\delta(a,r)$ is the δ-neighborhood of the sphere $C(a,r)$ centered at a with radius r.

Definition 7.1.4. *For $f \in L^1_{loc}(\mathbb{R}^n)$, define the maximal function $M_\delta f : [\frac{1}{2}, 2] \to \mathbb{R}$ by*

$$M_\delta f(r) = \sup_{a \in \mathbb{R}^n} \frac{1}{|C_\delta(a,r)|} \int_{C_\delta(a,r)} |f(y)| dy, \tag{7.1}$$

where $|E|$ denotes the n-dimensional Lebesgue measure on the set E.

As before, we would like to prove $L^p \to L^q$ estimates on the norm of this maximal operator and so obtain bounds on the Hausdorff dimension of circular Kakeya sets in $\mathbb{R}^n$. We have a predictible proposition that allows us to do precisely this.

Proposition 7.1.5. *Let A be a circular Kakeya set in $\mathbb{R}^n$. Suppose we have an estimate of the form*

$$\forall\ \epsilon > 0\ \exists\ C_\epsilon : ||M_\delta f||_{L^q([\frac{1}{2},2])} \leq C_\epsilon \delta^{-\alpha-\epsilon} ||f||_{L^p(\mathbb{R}^n)}, \tag{7.2}$$

for some $\alpha \geq 0$, $p, q < \infty$. Then $\dim_H(A) \geq n - p\alpha$.

The proof of this proposition is nearly identical to the proof of Proposition 2.3.1 once we know that it is sufficient to prove a bound on the function $M_\delta f$ when it is only defined over the interval $[\frac{1}{2}, 2]$, rather than on the function $M'_\delta f$ defined in the analogous way, only over the interval $(0, \infty)$. After all, we are ultimately after Hausdorff bounds on sets that contain a sphere of *every* radius, so we need to make sure it is sufficient to just consider spheres with radii in the interval $[\frac{1}{2}, 2]$. We refer the reader to Kolasa and Wolff's paper, Theorem 1$''$, for more discussion on how to proceed. We remark that Proposition 7.1.5 can be strengthened to assert that if A is a Borel set of $\mathbb{R}^n$ containing a positive $(n-1)$-dimensional Lebesgue measure subset of a sphere of radius r for some positive 1-dimensional Lebesgue measure set of r, then $\dim_H(A) \geq n - p\alpha$ if (7.2) holds.

In any case, taking the particulars of the proof of Proposition 7.1.5 for granted, we can now deduce Theorem 7.1.2 from the following stronger maximal function result.

Theorem 7.1.6. *If $n \geq 3$, then*

$$||M_\delta f||_{L^2([\frac{1}{2},2])} \lesssim \sqrt{\log \frac{1}{\delta}}||f||_{L^2(\mathbb{R}^n)}.$$

Just as with any Kakeya estimate, we will need to take advantage of a key geometrical fact. The following will look very similar to Córdoba's estimate on the size of the intersection of two tubes in $\mathbb{R}^n$ and indeed, we will use this fact in exactly the same way as we used Córdoba's.

Lemma 7.1.7. *Let $r, s \in [\frac{1}{2}, 2]$, $a, b \in \mathbb{R}^n$. If $n = 2$, then*

$$|C_\delta(a, r) \cap C_\delta(b, s)| \lesssim \frac{\delta^{3/2}}{(|r-s| + \delta)^{1/2}}, \tag{7.3}$$

and if $n \geq 3$, then

$$|C_\delta(a, r) \cap C_\delta(b, s)| \lesssim \frac{\delta^2}{|r-s| + \delta}. \tag{7.4}$$

Proof. We start with (7.3). Clearly we may assume the intersection is nonempty, else there is nothing to show. Now

$$|C_\delta(a, r) \cap C_\delta(b, s)| \leq |C_\delta(a, r)| \sim \delta.$$

Also, we trivially have $|r-s| + \delta \lesssim \delta$ if $|r-s| \lesssim \delta$, so

$$|C_\delta(a, r) \cap C_\delta(b, s)| \lesssim \frac{\delta^{3/2}}{\delta^{1/2}} \lesssim \frac{\delta^{3/2}}{(|r-s| + \delta)^{1/2}},$$

as desired. Now if $|r-s| \gtrsim \delta$, then we are in one of two cases: either the circular regions are essentially tangent or they are essentially transverse. More precisely, the intersection must be either connected or disconnected in two identical pieces as in the figure below.

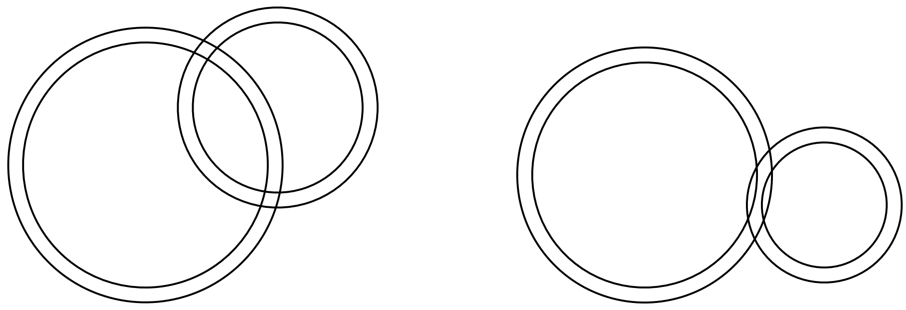

Figure 7.1: *Examples of two essentially transverse and two essentially tangent δ-thickened circles.*

Let x_1 and x_2 be the points in the plane where the two circles $C(a,r)$ and $C(b,s)$ intersect (with the understanding that $x_1 = x_2$ if the circles are exactly tangent).

If $|x_1 - x_2| \geq 10\delta$, then we are in the transverse case, meaning each piece of the intersection can be contained in a $\sim \delta$-square. Clearly the intersection consists of two similar pieces and the larger the distance $|x_1 - x_2|$, the more exactly like δ-squares these pieces become. If we are in the extreme case where $|x_1 - x_2| = 10\delta$, then we still have that each piece of the intersection must have diameter $\lesssim 5\delta$ to ensure that the pieces are indeed separate. Consequently, we have

$$|C_\delta(a,r) \cap C_\delta(b,s)| \lesssim \delta^2 < \delta^{3/2} \lesssim \frac{\delta^{3/2}}{(|r-s|+\delta)^{1/2}}, \tag{7.5}$$

since $|r-s|+\delta \leq 2$ always.

If instead $|x_1 - x_2| \leq 10\delta$, then the two δ-thickened circles are essentially tangent. To make the analysis clear, we are going to contain the intersection $C_\delta(a,r) \cap C_\delta(b,s)$ in a larger intersection of circular neighborhoods corresponding to new circles that are exactly tangent.

By applying a translation, we can always make one of our circles centered at the origin; we will take $b = 0$. Also, we may assume that $s \leq r$. Now let m denote the midpoint on the line segment connecting x_1 and x_2, so that $|0 - m| = s - \lambda_1$, $|a - m| = r - \lambda_2$, where λ_1, λ_2 are (presumably) small; see Figure 7.2 on the following page. Clearly, we have the following containments:

$$C_\delta(a,r) \subset C_{\delta+\lambda_2}(a, r-\lambda_2), \quad \text{and} \quad C_\delta(0,s) \subset C_{\delta+\lambda_1}(0, s-\lambda_1). \tag{7.6}$$

We would like to assert that, in fact, we have $\lambda_1, \lambda_2 \lesssim \delta$. To verify this, consider the region in the plane between the circles $C(a,r)$ and $C(0,s)$. What we see then is that this region is precisely the disjoint union of two circular segments, one corresponding to the sector of the circle $C(a,r)$ between x_1 and x_2, and the other corresponding to the sector of the circle $C(0,s)$ between x_1 and x_2. [fig] Let θ_r, θ_s denote the defining angles of these respective circular sectors. Clearly then, we have that the "width" of this region should be the sum of the heights of these circular segments, i.e.

$$\lambda_1 + \lambda_2 = s(1 - \cos\frac{\theta_s}{2}) + r(1 - \cos\frac{\theta_r}{2}) \tag{7.7}$$

using a well known formula from basic plane geometry. Examining the figure on the next page, we see that

$$\begin{aligned} \cos\frac{\theta_s}{2} &= \sqrt{s^2 - \left(\frac{|x-y|}{2}\right)^2} \div s \\ &\gtrsim \sqrt{1-\delta^2} \\ &= \sqrt{(1-\delta)(1+\delta)} \\ &\geq 1 - \delta. \end{aligned}$$

The same calculation for $\cos\frac{\theta_r}{2}$ shows then that we can bound (7.7) by $\lesssim \delta$; therefore, $\lambda_1, \lambda_2 \lesssim \delta$ and (7.6) becomes

$$C_\delta(a,r) \subset C_{C\delta}(a,r'), \quad \text{and} \quad C_\delta(0,s) \subset C_{C\delta}(0,s'),$$

for some constant C. Notice that we still have $|r' - s'| \gtrsim \delta$. So to understand the essentially tangential case, it suffices to analyze the case where our circles are exactly tangent.

Working within this new framework, we see that the intersection of our new circular neighborhoods is automatically a nice, single connected set. Now the width of this set in the direction of the normal to the circles must be $\lesssim C\delta$. So if we can estimate the diameter of this set in the perpendicular direction, then we should have a good estimate on the size of the intersection. Clearly, this diameter will join the points of intersection of the circles $C(a, r' + C\delta)$ and $C(0, s' + C\delta)$. We can estimate the length of this line segment then by the arc length of the corresponding circular sector. The estimate is independent of the choice of circle, so we opt to work with $C(0, s' + C\delta)$.

Let l denote the arc length defined above and let θ denote the angle of the corresponding circular sector in $C(0, s' + C\delta)$. Utilizing basic trigonometry again, we find that $\cos\frac{\theta}{2} = \frac{s'}{s'+C\delta}$ since our defining circles are exactly tangent. Thus,

$$\theta \sim \sqrt{1 - \cos\theta} \sim \sqrt{\frac{C\delta}{s' + C\delta}}.$$

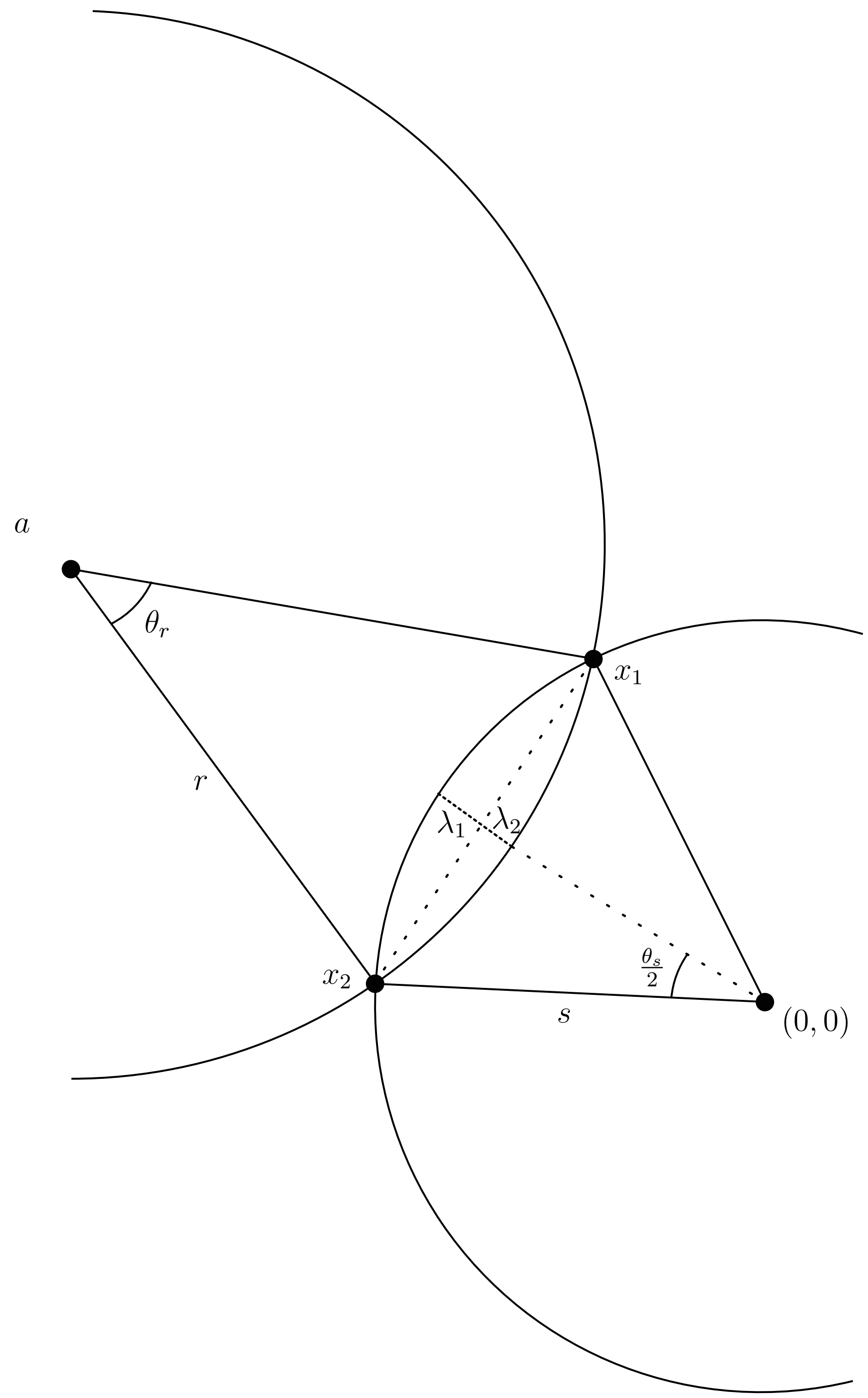

Figure 7.2: *Close-up of the region in the plane between two essentially tangent circles. Notice that the "height" of this region is* $|x_1 - x_2|$ *while the "width" is* $\lambda_1 + \lambda_2$*, the sum of the heights of the circular segments defined by* θ_s *and* θ_r*.*

Applying the well known formula for arc length, we arrive at the estimate

$$\begin{aligned} l &= \theta(s' + C\delta) \\ &\sim \sqrt{C\delta(s' + C\delta)} \\ &\lesssim \sqrt{\delta}. \end{aligned}$$

So we have the estimate $|C_\delta(a, r) \cap C_\delta(0, s)| \lesssim \delta^{3/2}$, which suffices as in (7.5).

For the second inequality (7.4), we can directly apply our work for (7.3). If $|r - s| \lesssim \delta$, then the exact reasoning will work as before. If $|r - s| \gtrsim \delta$, then again we consider the transverse and tangential cases separately. For the former, we apply the same reasoning to find that the intersection can be contained in a $\sim (\delta \times \delta \times 1 \times \cdots \times 1)$-box and so the estimate follows immediately. If the spheres are nearly tangent, then they essentially share a normal vector and we can execute the same containment trick as before in order to simplify to the case when the spheres are exactly tangent. As before, in this normal direction the intersection has length at most 2δ. In any orthogonal direction, we see that the intersection has length $\sim l$ where l is defined as before. Thus we pick up an extra factor of $\sqrt{\delta}$ in each of these $(n - 1)$-orthogonal directions and we have

$$|C_\delta(a, r) \cap C_\delta(b, s)| \lesssim \delta^{\frac{n+1}{2}} \leq \delta^2$$

for all $n \geq 3$. (7.4) follows. □

We can now prove the main result of this section.

Proof of Theorem 7.1.6. We begin by discretizing our domain. If $|r - s| < \delta$, then there exists some $x \in \mathbb{R}^n$ such that

$$\begin{aligned} M_\delta f(s) &\leq \frac{2}{|C_\delta(x, s)|} \int_{C_\delta(x,s)} f(y)dy \\ &\lesssim \frac{1}{|C_{2\delta}(x, r)|} \int_{C_{2\delta}(x,r)} f(y)dy \\ &\leq M_{2\delta} f(r), \end{aligned}$$

since $C_\delta(x, s) \subset C_{2\delta}(x, r)$. For every nonnegative integer $j \leq \frac{3}{2\delta}$, let $r_j = \frac{1}{2} + j\delta$. Then

$$\begin{aligned} ||M_\delta f||^2_{L^2([\frac{1}{2},2])} &= \int_{\frac{1}{2}}^{2} M_\delta f(r)^2 dr \\ &\leq \sum_{j=0}^{\frac{3}{2\delta}} \int_{\frac{1}{2}+j\delta}^{\frac{1}{2}+(j+1)\delta} M_\delta f(r)^2 dr \end{aligned}$$

$$\lesssim \sum_{j=0}^{\frac{3}{2\delta}} M_\delta f(r_j)^2 \int_{\frac{1}{2}+j\delta}^{\frac{1}{2}+(j+1)\delta} dr$$
$$= \sum_{j=0}^{\frac{3}{2\delta}} \delta M_\delta f(r_j)^2;$$

thus, it suffices to show the estimate

$$\left(\sum_{j=0}^{\frac{3}{2\delta}} \delta M_\delta f(r_j)^2 \right)^{1/2} \lesssim \sqrt{\log \frac{1}{\delta}} ||f||_{L^2(\mathbb{R}^n)}.$$

Now we apply duality: write

$$\left(\sum_j \delta M_\delta f(r_j)^2 \right)^{1/2} = \sum_j \delta y_j M_\delta f(r_j)$$
$$\lesssim \sum_j \frac{\delta y_j}{|C_\delta(x_j, r_j)|} \int_{C_\delta(x_j,r_j)} f(y) dy$$

for an appropriate choice of points $x_j \in \mathbb{R}^n$ and sequence $\{y_j\}$ with $\sum_j \delta |y_j|^2 = 1$. Define

$$b_j = \frac{\delta y_j}{|C_\delta(x_j, r_j)|}$$

and notice that $b_j \sim y_j$ since $|C_\delta(x_j, r_j)| \sim \delta$. Applying Cauchy-Schwarz, we see that we have

$$\left(\sum_j \delta M_\delta f(r_j)^2 \right)^{1/2} \lesssim || \sum_j b_j \chi_{C_\delta(x_j,r_j)} ||_2 \cdot ||f||_2;$$

therefore, it suffices to show that

$$|| \sum_j b_j \chi_{C_\delta(x_j,r_j)} ||_2^2 \lesssim \log \frac{1}{\delta}. \tag{7.8}$$

For notational convenience, let $C_j = C_\delta(x_j, r_j)$. Now

$$|| \sum_j b_j \chi_{C_j} ||_2^2 = \int | \sum_{j,k} b_j b_k \chi_{C_j}(y) \chi_{C_k}(y) | dy$$
$$\leq \sum_{j,k} |b_j||b_k| \int \chi_{C_j \cap C_k}(y) dy$$

by the triangle inequality. For notational convenience again, we can now assume that b_j is a positive sequence. Thus, applying our geometrical fact, Lemma 7.1.7, we find that this expression is dominated by

$$\sum_{j,k} b_j b_k \frac{\delta^2}{|r_j - r_k| + \delta} = \sum_{j,k} b_j b_k \frac{\delta}{|j-k|+1}.$$

We rewrite the above as

$$\sum_{j,k} \frac{b_j\sqrt{\delta}}{(|j-k|+1)^{1/2}} \cdot \frac{b_k\sqrt{\delta}}{(|j-k|+1)^{1/2}}$$

and apply Cauchy-Schwarz again to find

$$\begin{aligned} ||\sum_j b_j \chi_{C_j}||_2^2 &\lesssim \left(\sum_{j,k} \delta b_j^2 \frac{1}{|j-k|+1}\right)^{1/2} \left(\sum_{j,k} \delta b_k^2 \frac{1}{|j-k|+1}\right)^{1/2} \\ &= \sum_{j,k} \delta b_j^2 \frac{1}{|j-k|+1} \\ &\lesssim \sum_j \delta b_j^2 \log \frac{1}{\delta} \\ &\lesssim \log \frac{1}{\delta}, \end{aligned}$$

verifying (7.8). □

Notice that we see now why we cannot readily expand this theorem to include the case when $n = 2$. Our Lemma 7.1.7 is not strong enough to provide a meaningful bound and unfortunately it cannot be easily improved. Indeed, we saw in the proof of this lemma that the factor of $\delta^{3/2}$ was required and this is not good enough to deduce an optimal result in $\mathbb{R}^2$ using the method of proof above. Kolasa and Wolff make a different argument to prove their Theorem 7.1.3; we will not go into detail here.

Clearly there are many similarities between the circular Kakeya problem and the classical one. Many now familiar reductions and tools are applicable in the new setting, although we see that the different geometry has of course changed the problem in unfamiliar ways as well. We mention briefly some other variations on the Kakeya problem here. All of the below references are taken from Wolff's survey paper [32].

It is natural to consider maximal functions given by suprema of averaging operators over δ-thickened curves instead of lines to attempt to bridge the gap between the classic and the circular Kakeya problems. Such operators are in many ways less well behaved than our endpoint operators though. See [7], [8], and [26] for more information.

The problem of $(n, 2)$ sets shares many similarities with Kakeya. This problem essentially asks if a set in $\mathbb{R}^n$ which contains a translate of every 2-plane must necessarily have positive measure. This is known to be true when $n = 3$ [25] and $n = 4$ [6]. Further results can be found in [1].

In the next section, we will study in detail the natural analogue of the Kakeya problem over finite fields. In this new setting, we actually have an optimal result in all dimensions. This extraordinary result is due to Dvir and has been known only for several years.

7.2 The Finite Field Kakeya Problem

Let $\mathbb{F}$ denote a finite field of q elements. For $n \in \mathbb{N}$, consider the vector space $\mathbb{F}^n$. This vector space bears many similarities to the analytically richer $\mathbb{R}^n$; in particular, we may speak of lines, planes and all the other customary submanifolds in this new vector space. For the Kakeya problem, we make the obvious definition.

Definition 7.2.1. *A set $K \subset \mathbb{F}^n$ is a* ***Kakeya set*** *if for every $x \in \mathbb{F}^n$ there exists a point $y \in \mathbb{F}^n$ such that the line*

$$L_{y,x} = \{y + a \cdot x \mid a \in \mathbb{F}\}$$

is contained in K.

So a Kakeya set in $\mathbb{F}^n$ is just any set that contains a line in every direction. In the notation above, for any "direction" $x \in \mathbb{F}^n$, we may find an "intercept" $y \in \mathbb{F}^n$ such that the unique line thus defined is contained in the set. Notice we do not have a concept of unit length here since there is no metric space structure imposed on the vector space $\mathbb{F}^n$.

The classical Kakeya conjecture in $\mathbb{R}^n$ asserts that Kakeya sets must have full Hausdorff dimension n. So even though these sets may have Lebesgue measure zero, they still fill out $\mathbb{R}^n$ enough to not lose anything in the way of dimension. In the finite field case, since we have no metric, we cannot construct a Hausdorff (or Lebesgue) measure. However, there is still a natural concept of dimensionality in this context. The size of $\mathbb{F}^n$ is exactly q^n, so any subset of full dimension must have cardinality a fixed proportion of this quantity, fixed in that it should not depend on the underlying size of the field $\mathbb{F}$. Evidently then, a subset of $\mathbb{F}^n$ with full dimension should have cardinality no less than $C_n \cdot q^n$ for some constant C_n depending only on n. This is exactly what Dvir proved for Kakeya sets in $\mathbb{F}^n$ and we will present his extremely beautiful and elementary proof after a brief discussion on the easy and useful relationship between monomials and binomial coefficients.

Let $m \geq k$ be integers; we claim that

$$\left(\frac{m}{k}\right)^k \leq \binom{m}{k} \leq \left(\frac{m \cdot e}{k}\right)^k,$$

where e is the Euler number. The first inequality is obvious from the fact that

$$\frac{m-j}{k-j} \geq \frac{m}{k}$$

for all $0 \leq j \leq k-1$. The second inequality follows by expanding e^k as a Taylor series and then estimating the sum from below by the kth term. Thus we have the relation $\binom{m}{k} \sim C_k \cdot m^k$, where C_k is a constant depending only on k.

Theorem 7.2.2. *Let $K \subset \mathbb{F}^n$ be a Kakeya set. Then*

$$|K| \geq C_n \cdot q^n,$$

where C_n depends only on n.

Proof. The proof is by contradiction. We will prove the theorem for any q and n, but to begin with we will fix n and choose q large enough. More precisely, let q be large enough so that $\binom{q+n-1}{n} \sim q^n$. This very particular choice of binomial coefficient may at first seem unnatural and indeed it is not completely necessary, but it will appear very naturally when we exploit the fact that binomial coefficients are powerful counting tools.

So with the value of n fixed and our choice of q sufficiently large, we suppose that K is a Kakeya set of size less than $\binom{q+n-1}{n}$. We claim that there exists a nonzero polynomial $P \in \mathbb{F}[x_1, \ldots, x_n]$ of degree at most $q-1$ such that $P(x) = 0$ for all $x \in K$. The existence of such a polynomial is the crux of the argument and we must be sure that it indeed exists before proceeding.

Let P be a generic polynomial in $\mathbb{F}[x_1, \ldots, x_n]$ of degree at most $q-1$. If we can find nontrivial coefficients for each term in P such that $P(x) = 0$ for all $x \in K$, then we will have what we are looking for. For each $x \in K$, write down the equation $P(x) = 0$ in standard form; thus we write down $|K| < \binom{q+n-1}{n}$ equations. We want to solve this system of equations for the unknown coefficients. In order to do this, we have to know how many unknowns we are dealing with.

There are exactly $\binom{q-1+n-1}{n-1}$ monomials in $\mathbb{F}[x_1, \ldots, x_n]$ of degree $q-1$ (the number of ways to place $q-1$ indistinguishable objects into n distinguishable boxes), thus there are no more than this many coefficients of the degree $q-1$ terms. Similarly, there are no more than $\binom{q-2+n-1}{n-1}$ coefficients of the degree $q-2$ terms and so on, all the way down to the coefficient of the constant term in P. Summing these up, we find that there are no more than

$$\sum_{k=0}^{q-1} \binom{k+n-1}{n-1} = \sum_{k=n-1}^{q+n-2} \binom{k}{n-1} = \binom{q+n-1}{n}$$

coefficients in total. This last equality is an easy identity that can be quickly derived from an appeal to Pascal's triangle.

We have just shown that the number of unknown coefficients is strictly larger than the number of constraints in our homogeneous system of linear equations. Thus a nontrivial solution exists and we pick one of these to produce our nonzero polynomial P that vanishes on the set K.

Now write $P = \sum_{i=0}^{q-1} P_i$, where P_i is the homogeneous part of degree i of P. Fix $y \in \mathbb{F}^n$. Then since K is a Kakeya set, there exists some $b \in \mathbb{F}^n$ such that $P(b + ay) = 0$ for all $a \in \mathbb{F}$. For fixed b and y this is a polynomial of degree $q-1$ in a which vanishes for all $a \in \mathbb{F}$. Since $|\mathbb{F}| = q$, this polynomial in a is identically zero and so all its coefficients are zero. In particular the coefficient of a^{q-1} is zero and we claim that this coefficient is exactly $P_{q-1}(y)$. To see this, write

$$P(x) = \sum_{i=0}^{q-1} \sum_{|\alpha|=i} c_\alpha x^\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $c_\alpha \in \mathbb{F}$. Then

$$P(b + ay) = \sum_{i=0}^{q-1} \sum_{|\alpha|=i} c_\alpha \prod_{j=1}^{n} (b_j + ay_j)^{\alpha_j}.$$

Now the coefficient of a^{q-1} in $\prod(b_j + ay_j)^{\alpha_j}$ is $\prod y_j^{\alpha_j}$ so the coefficient of a^{q-1} in $P(b+ay)$ is

$$\sum_{|\alpha|=q-1} c_\alpha y^\alpha = P_{q-1}(y),$$

verifying the claim.

Thus we have that P_{q-1} is identically zero. So this must mean that $P = \sum_{i=0}^{q-2} P_i$ and we can repeat the above argument to conclude that the polynomials $P_{q-2}, P_{q-3}, \ldots, P_1$ are all identically zero. Hence $P = P_0$, the constant term. But P_0 must be zero since P vanishes at some point (in particular, at any point in K). So P is the zero polynomial and we arrive at a contradiction.

Recall that the above was all done supposing that q is large enough with respect to a fixed n. To complete the proof, just notice that there are finitely many q smaller than n, so we may directly compute a constant C_n that satisfies $|K| \geq C_n \cdot q^n$ in these cases. Taking the minimum of these constants and the one above in the case of q large enough, we thus have the theorem for any fixed $n \in \mathbb{N}$. □

The stark contrast between the powerful simplicity of Dvir's resolution of the Kakeya conjecture in $\mathbb{F}^n$ and the complicated machinery required to derive our partial Kakeya estimates in $\mathbb{R}^n$ should serve to remind the reader of the vast differences between the analytical nature of finite fields and the classical Euclidean one. The proof that we have given above may indeed be beautifully simple, but it still required the diligent work of many great minds over the course of many years to be presented in such a neat fashion. In particular, it should be noted that Dvir's original result was that Kakeya sets in $\mathbb{F}^n$ must have size no less than $C_n \cdot q^{n-1}$. Following the initial publication of this result, Alon and Tao observed through private communications with Dvir that his proof could be modified to give the optimal bound that we have shown.

It was hoped for many years that deeper insight into the Kakeya problem over finite fields would provide a richer understanding of the classical problem over $\mathbb{R}^n$. The one downside of the above result then is that we seem to have pushed the analysis to the optimal level without gaining this extra insight. But perhaps we should not be so hasty to make this statement as we possibly have just not had the epiphany yet to help us bridge the gap.

7.3 Other Conjectures and Problems Related to Kakeya

In this final section, we aim to outline the Kakeya problem's place in the hierarchy of other classical problems in harmonic analysis. We will state results and give references without concentrating on proofs here since the literature is already quite extensive.

Much of the following can be found in [27], in addition to proofs and further results. We begin with the restriction problem. This problem attempts to address the "continuity" (not in the technical calculus sense) of the Fourier transform in $\mathbb{R}^n$. More precisely, we know that the Fourier transform of an $L^1(\mathbb{R}^n)$ function is continuous by the Riemann-Lebesgue Lemma (see [16]); thus, it is also defined everywhere on $\mathbb{R}^n$. However, by Plancherel's Theorem, the Fourier transform of an L^2 function is again L^2; thus, it is only defined almost everywhere on $\mathbb{R}^n$. Why this is a concern is that in $n \geq 2$, there are many natural objects that we would like to restrict the Fourier transform to, in particular, we would like to restrict the Fourier transform of an $L^p(\mathbb{R}^n)$ function to the unit sphere. But this object has n-dimensional Lebesgue measure zero in $\mathbb{R}^n$, so we cannot even well-define the transform on this object by simply appealing to some old, classic result. Progress can still be made though by making the notion of restriction precise. Although the following can be formulated for any smooth submanifold of $\mathbb{R}^n$, we restrict our attention to the canonical case of the unit sphere.

Definition 7.3.1. *We say that the* **L^p** ***restriction property****, denoted by $R(p)$, holds if there exists a $q = q(p)$ so that*

$$||\hat{f}||_{L^q(S^{n-1})} \leq C_{p,q,n}||f||_{L^p(\mathbb{R}^n)} \tag{7.9}$$

holds for every $f \in L^p$, where $\hat{f}$ is the customary Fourier transform of f in $\mathbb{R}^n$.

The first formidable restriction result is the theorem of Stein and Tomas which asserts that $R(p)$ holds whenever $1 \leq p \leq \frac{2n+2}{n+3}$ with $q = 2$. It is also known that no restriction theorem can hold for $p \geq \frac{2n}{n+1}$. This fact coupled with the fact that $R(p)$ holds for all $1 \leq p < \frac{2n}{n+1}$ in $\mathbb{R}^2$, leads us to a natural conjecture.

Conjecture 7.3.2 (Restriction Conjecture)**.** *For $n \geq 3$, $R(p)$ holds for all $1 \leq p < \frac{2n}{n+1}$.*

Many improvements have been made to the benchmark Stein-Tomas restriction theorem but we are still far from the full resolution of the restriction conjecture. Much of this progress has relied on analyzing a local analogue of the restriction problem; this program of study was initiated by Bourgain [6]. Let $R(p, \alpha)$ denote the localized restriction estimate

$$||\hat{f}||_{L^q(S^{n-1})} \leq C_{p,q,n} r^{\alpha} ||f||_{L^p(B(0,r))}$$

for some $q = q(p)$. Similar in spirit to Kakeya, it suffices to consider only diagonal estimates $q = p$ in the context of the full restriction conjecture. Notice that $R(p, 0)$ is equivalent to $R(p)$ and that global restriction estimates imply local restriction estimates. Bourgain showed that this implication can be partially reversed, namely,

Proposition 7.3.3 (Bourgain)**.** *If $p \leq \frac{2n+2}{n+3}$, $0 \leq \alpha \leq \frac{n+1}{2p'}$, then the local restriction estimate $R(p, \alpha)$ implies the global restriction estimate $R(q)$ whenever*

$$q' \geq 2 + \frac{1}{\frac{n+1}{2p'} - \alpha}.$$

So we can prove global restriction theorems by deriving local estimates. Of course the question now becomes how to go about proving local restriction estimates and this is where Kakeya enters the framework. Bourgain used his Kakeya estimate $\mathcal{K}(\frac{n+1}{2})$ to show a corresponding local restriction estimate. More specifically, we have

Proposition 7.3.4 (Bourgain)**.** *Suppose we have a Kakeya estimate $\mathcal{K}(p)$. Then we have a corresponding local restriction estimate $R(p, \alpha)$ for $\alpha = \frac{n+1}{8p'} - \frac{n-1}{8} + \epsilon$ for any $\epsilon > 0$.*

Clearly, coupling this proposition with Proposition 7.3.3, we see that a Kakeya estimate in fact implies a restriction estimate. Bourgain's argument gave an improvement to Stein-Tomas by increasing the range of p; in $\mathbb{R}^3$, this improves the Stein-Tomas range of $1 \leq p \leq \frac{4}{3}$ to $1 \leq p \leq \frac{58}{43}$. Unfortunately, this result is still far from the conjectured optimal one, corresponding to the range $1 \leq p < 3$ in $\mathbb{R}^3$. See [6] or [29] for details.

One final thing to note is that in fact global restriction estimates imply Kakeya estimates and that a full resolution of the restriction conjecture would settle the Kakeya conjecture. It is not known whether there is actually a formal equivalence between the two conjectures, only the partial equivalence outlined above.

The restriction problem is closely related to two other classical problems in harmonic analysis, that of the L^p boundedness of the Bochner-Riesz means and that of local smoothing of the wave equation. Consequently, we see that the Kakeya problem is also related to these problems.

Definition 7.3.5. *The **Bochner-Riesz spherical summation operator** of order δ is given by the formula*

$$S_R^\delta f(x) = \int_{|\xi|\leq R} \hat{f}(\xi) \left(1 - \frac{|\xi|^2}{R^2}\right)^\delta e^{2\pi i x\cdot\xi} d\xi$$

for $f \in L^p(\mathbb{R}^n)$.

In one dimension, S_R^δ is a bounded $L^p \to L^p$ operator for all $1 < p < \infty$ and in fact $f(x) = \lim_{R\to\infty} S_R^\delta f(x)$. This is intimately related to the fact that the Fourier series of an arbitrary $L^p(\mathbb{R})$ function f converges almost everywhere to the function f for any $1 < p < \infty$. This follows from the theorem of Carleson-Hunt-Sjölin and the transference principle (see [17]). In $\mathbb{R}^n$, $n \geq 2$, much less is known. Let $BR(p, \alpha)$ denote that $S_1^{\delta(p)+\alpha}$ is bounded on $L^p(\mathbb{R}^n)$ where $\delta(p) = \max\{n|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}, 0\}$. Then we have the following conjecture.

Conjecture 7.3.6 (Bochner-Riesz Conjecture)**.** *If $n \geq 2$, $1 \leq p \leq \infty$, and $\epsilon > 0$, then $BR(p, \epsilon)$ holds.*

We have the amazing result due to Tao [30] that the Bochner-Riesz conjecture implies the restriction conjecture. This result is very similar in flavor to his theorem that the Kakeya and Nikodym maximal function conjectures are equivalent (see Chapter 3).

These problems are all related to another classic problem in PDE's, that of local smoothing of the wave equation. The problem, stated in one of its simplist forms, is as follows.

Conjecture 7.3.7 (Local Smoothing of the Wave Equation)**.** *If $u(x, t)$ is a solution of the homogeneous Cauchy problem*

$$\left(-\frac{\partial^2}{\partial t^2} + \Delta\right) u(x, t) = 0,$$
$$u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

then one has

$$||u||_{L^p(\mathbb{R}^n\times[1,2])} \lesssim ||(1 + \sqrt{-\Delta})^\epsilon f||_p$$

for all $\epsilon > 0$, where $p = \frac{2n}{n-1}$ and Δ is the Laplacian.

This conjecture has been shown to be formally weaker than all the above conjectures and so would imply all of them. See [32], [33] for more information.

In summary, we have the following table of implications.

	Local Smoothing	
	$\Downarrow$	
	Bochner-Riesz	
	$\Downarrow$	
	Restriction	
	$\Downarrow$	
Maximal Kakeya	$\Leftrightarrow$	Maximal Nikodym
$\Downarrow$		$\Downarrow$
Kakeya Set		Nikodym Set

Table 7.1: Known Implications between Bochner-Riesz, Restriction and Kakeya

Finally, we remark that the Kakeya problem can be viewed from a more purely combinatorial perspective, akin to the argument of Katz and Tao [21] establishing the estimate $\mathcal{K}(\frac{4n+3}{7})$. See [24] for a thorough discussion.

Bibliography

[1] D. Alvarez: *Bounds for some Kakeya-type Maximal Functions*, Berkeley thesis, (1997).

[2] C. Bennett and R. Sharpley: *Interpolation of Operators*, Academic Press, Inc. (1988).

[3] A.S. Besicovitch: *Sur Deux Questions de l'Intégrabilité des fonctions*, J. Société des Math. et de Phys. à l'Univ. à Perm, **2** (1919), 105-123.

[4] A.S. Besicovitch: *On Kakeya's Problem and a Similar One*, Math. Zeitschrift **27** (1928), 312-320.

[5] A.S. Besivovitch and R. Rado: *A Plane Set of Measure Zero Containing Circumferences of Every Radius*, J. London Math. Soc., **43**, (1968), 717-719.

[6] J. Bourgain: *Besicovitch Type Maximal Operators and Applications to Fourier Analysis*, Geometric and Functional Analysis **1** (1991), 147-187.

[7] J. Bourgain *L^p Estimates for Oscillatory Integrals in Several Variables*, Geometric and Functional Analysis **1** (1991), 321-374.

[8] J. Bourgain *Some New Estimates for Oscillatory Integrals*, in Essays on Fourier Analysis in Honor of Elias M. Stein, ed. C. Fefferman, R. Fefferman, S. Wainger, Priceton University Press (1994).

[9] A. Carbery: *Restriction Implies Bochner-Riesz for Paraboloids*, Math. Proc. Camb. Phil. Soc. **111** (1992), 525-529.

[10] J.B. Conway: *A Course in Functional Analysis*, Springer, Second Edition (2007).

[11] A. Córdoba: *The Kakeya Maximal Function and the Spherical Summation Multipliers*, American Journal of Mathematics **99** (1977), 1-22.

[12] R.O. Davies: *Some Remarks on the Kakeya Problem*, Proc. Camb. Phil. Soc. **69** (1971), 417-421.

[13] Z. Dvir: *On the Size of Kakeya Sets in Finite Fields*, J. Amer. Math. Soc. **22** (2009), 1093-1097.

[14] K. Falconer: *Fractal Geometry - Mathematical Foundations and Applications*, John Wiley & Sons, Second Edition (2003).

[15] C. Fefferman: *A Note on Spherical Summation Multipliers*, Israeli J. Math. **15** (1973), 44-52.

[16] G.B. Folland: *Real Analysis: Modern Techniques and their Applications*, John Wiley & Sons, Second Edition (1999).

[17] L. Grafakos: *Modern Fourier Analysis*, Springer, Second Edition (2009).

[18] J.R. Kinney: *A Thin Set of Circles*, Amer. Math. Monthly **75** (1968), 1077-1081.

[19] S. Kakeya: *Some Problems on Maximum and Minimum Regarding Ovals*, Tôhoku Science Reports **6** (1917), 71-88.

[20] N.H. Katz, I. Łaba and T. Tao: *An Improved Bound on the Minkowski Dimension of Besicovitch Sets in* $\mathbb{R}^3$, Ann. of Math. (2) **152** (2000), 383-446.

[21] N.H. Katz and T. Tao: *New Bounds for Kakeya Problems*, J. Anal. Math. **87** (2002), 231-263.

[22] N.H. Katz and T. Tao: *Recent Progress on the Kakeya Conjecture*, Publ. Mat. (2002), 161-179.

[23] L. Kolasa and T. Wolff: *On Some Variants of the Kakeya Problem*, Pacific J. Math. **190** (1999), 111-154.

[24] I. Łaba: *From Harmonic Analysis to Arithmetic Combinatorics*, Bull. Amer. Math. Soc. **45** (2008), 77-115.

[25] J.M. Marstrand: *Packing Planes in* $\mathbb{R}^3$, Mathematika **26** (1979), 180-183.

[26] C. Sogge: *Concerning Nikodym-type Sets in* 3*-dimensional Curved Spaces*, J. Amer. Math. Soc. **12** (1999), 1-31.

[27] E. Stein: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press (1993)

[28] E. Stein and G. Weiss: *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press (1971)

[29] T. Tao: *Lecture Notes*, available at math.ucla.edu/ tao/254b.1.99s/ (1999).

[30] T. Tao: *The Bochner-Riesz Conjecture Implies the Restriction Conjecture*, Duke Math. J. **96** (1999), 363-375.

[31] T. Wolff: *An Improved Bound for Kakeya Type Maximal Functions*, Rev. Mat. Iberoamericana **11** (1995), 651-674.

[32] T. Wolff: *Recent Work Connected with the Kakeya Problem*, Prospects in Mathematics, Princeton (1996), 129-162.

[33] T. Wolff: *Lectures on Harmonic Analysis*, University Lecture Series **29** (2003).